Diagonal free field matrix correlators, global symmetries and giant gravitons

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ABSTRACT: We obtain a basis of diagonal free field multi-matrix 2-point correlators in a theory with global symmetry group G. The operators fall into irreducible representations of G. This applies for gauge group $\mathrm{U}(N)$ at finite N. For composites made of n fundamental fields, this is expressed in terms of Clebsch-Gordan coefficients for the decomposition of the n-fold tensor products of the fundamental field representation in terms of $G \times S_n$ representations. We use this general construction in the case of the SL(2) sector of $\mathcal{N} = 4$ SYM. In this case, by using oscillator constructions, we reduce the computation of the relevant Clebsch-Gordans coupling infinite dimensional discrete series irreps of SL(2) to a problem in symmetric groups. Applying these constructions we write down gauge invariant operators with a Fock space structure similar to that arising in a large angular momentum limit of worldvolume excitations of giant gravitons. The Fock space structure emerges from Clebsch multiplicities of tensor products of symmetric group representations. We also give the action of the 1-loop dilatation operator of $\mathcal{N} = 4$ SYM on this basis of multi-matrix operators.

Contents

1   Introduction

AdS/CFT [1–3] allows a translation of many hard questions of quantum gravity to questions in the dual CFT. Intriguing lessons on quantum gravity have taken the form of the stringy exclusion principle [4]. Some aspects of this principle find a geometrical expression in the properties of giant gravitons [5, 6]. The systematic study of the correlation functions of gauge invariant multi-trace local operators in the half-BPS sector has been fruitful in identifying CFT duals [7–9] of giant gravitons, strings attached to them [10–12] as well as bulk geometries resulting from their back-reaction on space-time [13]. An important step in these maps to spacetime is the identification of appropriate diagonal bases in the space of multi-trace operators. For holomorphic operators constructed from a single complex matrix, relevant to the half-BPS sector, this was solved in [8]. Recent generalizations including non-holomorphic operators and multi-matrices have been achieved [14–16].

In [15] we solved the diagonalisation in the sector of holomorphic operators constructed from M complex matrices transforming in the fundamental of U(M) (the case of $M = 3$ being of interest in $\mathcal{N} = 4$ SYM). We gave covariant operators with correlators of the simple form

$$\left\langle (\hat{\mathcal{O}}^{\Lambda, M_\Lambda, i})^I_J (\hat{\mathcal{O}}^{\dagger \Lambda', M'_{\Lambda'}, i'})^K_L \right\rangle = \delta^{\Lambda\Lambda'} \delta^{M_\Lambda M'_{\Lambda'}} \sum_{\sigma \in S_n} D^\Lambda_{ii'}(\sigma) \ (\sigma)^K_J (\sigma^{-1})^I_L \qquad (1.1)$$

We will call this the *canonical covariant form*. Λ is the U(M) irreducible representation and M_Λ labels the state within this irrep. $D^\Lambda_{ii'}(\sigma)$ is the orthogonal matrix representation of the S_n representation Λ. There is no spacetime dependence because we are considering a 4d analogue of the Zamolodchikov metric used in 2d CFT. We will often use the expression 'two-point function' interchangeably with this metric.

These covariant operators lead to gauge invariant operators with diagonal correlation functions

$$\left\langle \mathcal{O}^{\Lambda, M_\Lambda, R, \tau} \mathcal{O}^{\dagger \Lambda', M'_{\Lambda'}, R', \tau'} \right\rangle \propto \delta^{\Lambda\Lambda'} \delta^{M_\Lambda M'_{\Lambda'}} \delta^{RR'} \delta^{\tau\tau'} \qquad (1.2)$$

R is the U(N) representation which organises the multi-trace structure of the operator.

In this paper we will explain how to get the canonical covariant form (1.1) for any global symmetry group G. This will use Schur-Weyl duality which we review below. The diagonal operators for the gauge invariant operators (1.2) follows automatically (see section 6).

Applied to $\mathcal{N}=4$ SYM this means we solve the problem of writing down a basis for the space of all gauge invariant operators in the theory (and in any subsector of the theory). Operators are labelled by lowest weight representations of the global symmetry group and diagonalise the free two-point correlation functions. We write these operators in terms of appropriate Clebsch-Gordon coefficients and it should be noted that finding the CG coefficients themselves is still a difficult problem in general, but we show how to do this in the particular example of the SL(2) sector.

1.1 Schur-Weyl duality and $G \times S_n$ Clebsch decompositions

Let V_F be the fundamental representation of U(M) (or $GL(M)$). Classical Schur-Weyl duality gives the decomposition of $V_F^{\otimes n}$ in terms of irreducible representations of U(M). It relies on the fact that in the algebra of linear operators acting $V_F^{\otimes n}$, i.e. in $\text{End}(V_F^{\otimes n})$, the maximal subalgebra which commutes with U(M) is exactly the group algebra of S_n. It gives the decomposition of $V_F^{\otimes n}$ under U(M) $\otimes$ S_n :

$$V_F^{\otimes n} = \bigoplus_{\Lambda \vdash n} V_\Lambda^{\text{U}(M)} \otimes V_\Lambda^{S_n} \tag{1.3}$$

Here $\Lambda \vdash n$ denotes the fact that Λ runs over partitions of n, which correspond to Young diagrams with row lengths $(r_1, r_2 \cdots)$ with $r_i \geq r_{i+1}$. According to (1.3), $V_F^{\otimes n}$ has a complete basis of states of the form $|\Lambda, M_\Lambda, m_\Lambda\rangle$, where M_Λ label states in the irrep. of U(M) corresponding to the Young diagram Λ and m_Λ label states in the irrep. of S_n corresponding to the same Young diagram. For $n > M$, the above has to be qualified with the constraint that $c_1(\Lambda) \leq M$, i.e. the maximal length of the first column of the Young diagram is M.

Schur-Weyl duality is a special case of the double centraliser theorem (or double commutant theorem) which gives a decomposition generalising (1.3) for any algebra acting on a vector space $\mathcal{W}$ (for a brief statement of the key relevant facts see section 1 of [17] or textbooks such as [18]). The decomposition is given in terms of the algebra of interest A and the commutant (centraliser) of its action in the vector space, often denoted as $End_A(\mathcal{W})$. Let G be the global symmetry group of a field theory. Let V_F be a representation of G formed by a set of fields in the theory. We will consider $\mathcal{W} = V_F^{\otimes n}$ and the algebra of interest is the universal enveloping algebra of the Lie algebra of G. Often G will be a subgroup of the full symmetry group of a gauge theory. In the case of $\mathcal{N} = 4$ SYM, if we are interested in the six hermitian scalars (without space-time derivatives), V_F is the fundamental of SO(6). If we write the six hermitian scalars as three complex ones, and only consider the holomorphic combinations, then the symmetry group of interest is U(3) and V_F is the fundamental 3-dimensional representation. This is the case we studied in [15] and is covered by classical Schur-Weyl duality. In this paper one of the main examples will involve $G = $ SL(2) and V_F will be an infinite dimensional discrete series representation corresponding to a single scalar field X and its derivatives with respect to one light-cone direction, $\partial X, \partial^2 X, \ldots$. More generally we have fields $W_m \in V_F$ where m may run over an

infinite dimensional vector space. In the SL(2) example we would have $W_m = \partial^m \Phi$. We will then consider the n-field composites

$$W_{m_1} \otimes W_{m_2} \otimes \cdots \otimes W_{m_n} \tag{1.4}$$

which transform in the representation $V_F^{\otimes n}$ of G. This space also has an action of the symmetry group S_n which permutes the n fields. This S_n action commutes with G, but does not provide the full commutant of G. Equivalently there is a non-trivial commutant algebra of $G \times S_n$, denoted by $\text{Com}(G \times S_n)$. By the double-commutant theorem, this will organise the multiplicities of $G \times S_n$ representations. We have

$$(V_F^G)^{\otimes n} = \bigoplus_{\Lambda, \Lambda_1} V_\Lambda^G \otimes V_{\Lambda_1}^{S_n} \otimes V_{\Lambda, \Lambda_1}^{\text{Com}(G \times S_n)} \tag{1.5}$$

where Λ is a representation of the global group G and Λ_1 is a representation of S_n. V_{Λ, Λ_1} gives the multiplicity with which Λ of G and Λ_1 of S_n appear together. It is a representation of $\text{Com}(G \times S_n)$. The explicit form of $\text{Com}(G \times S_n)$ will not be needed in this paper, with V_{Λ, Λ_1} appearing simply as the carrier space of a multiplicity label. In diagonalising the free-field two point functions, the decomposition of $V_F^{\otimes n}$ in terms of the group $G \times S_n$ will be crucial.

The study of the multiplicity of the decomposition of n-fold tensor powers in terms of $G \times S_n$ is called the plethysm problem. This is solved for SU(2) in [19] and is, for more general G, the subject of a large mathematical literature. For developments on the use of the combinatorics related to plethysms in the context of chiral rings of a large class of $\mathcal{N} = 1$ SYM theories see [20]. For our purposes, we will be interested, not only in the multiplicities of plethysms but also the corresponding Clebsch-Gordan coefficients. Using these we will obtain explicit gauge invariant operators which diagonalize the Zamolodchikov metric.

1.2 Outline of paper and main results

In section 2 we will find the canonical covariant form for the SL(2) sector. In particular we will have in mind derivatives acting on one complex scalar X. In section 3, we will give explicit formulae for the SL(2) $\times S_n$ multiplicities involved. We will be heavily using the oscillator construction of SL(2). In section 4, we show, for any global symmetry G and for n-field composites with the basic fields transforming in any representation V of G, that the covariant canonical form follows once we construct n-field composites using the Clebsch-Gordan coefficients for the $G \times S_n$ decomposition of $V^{\otimes n}$. In section 5 we give various examples of this general case explaining how the diagonalisations in the U(M) and SL(2) sectors follow this general pattern, and how the canonical covariant corresponding to representations of the higher spin group $HS(1,1)$ are also included. We describe some useful facts concerning SO(6) $\times S_n$ multiplicities relevant to the SO(6) sector of six hermitian scalars and finally we show how to apply the construction of section 4 to the case where where G is of the product form $G_1 \times G_2$. In section 6 we review how to get the diagonal gauge-invariant operators from the canonical covariant form and show the compatibility of the counting with matrix model methods of [21]. Section 7 shows how our results on

the SL(2) sector can be used to provide the gauge theory operators dual to the worldvolume excitations of giants considered in [22]. The one-loop mixing of the gauge-invariant operators constructed in section 6 for $\mathcal{N} = 4$ is analysed in section 8.

2 SL(2) sector: covariant operators

We consider the SL(2) sector which we can view as a reduction of $\mathcal{N} = 4$ SYM to a sector with a single light-cone derivative of the complex scalar X. We choose $\partial \equiv (\partial_0 + \partial_3)/2$.

We find the basic two-point function

$$\left\langle \partial^{k_1} X^{\dagger i}_{\ j}(x)\, \partial^{k_2} X^k_l(0) \right\rangle = \frac{(-1)^{k_1}(k_1 + k_2 + 1)!}{x^{2+k_1+k_2}}\, \delta^i_l\, \delta^k_j \tag{2.1}$$

If we consider $\mathcal{N} = 4$ SYM on $\mathcal{R}^4$, we take our two operators to zero and infinity (corresponding to opposite poles of the conformally equivalent S^4) we have

$$\left\langle \partial^{k_1} X^{\dagger\prime i}_{\ j}(x' = 0)\, \partial^{k_2} X^k_l(x = 0) \right\rangle = \delta^{k_1 k_2}(k_1!)^2\, \delta^i_l\, \delta^k_j \tag{2.2}$$

where $x' = x/x^2$ is the coordinate patch around the north pole and x around the south. This technique is well known from the studies of conformal field theories in two dimensions and the above is known as the Zamolodchikov metric (see [23, 24] for a general account and [25] for applications to $\mathcal{N} = 4$ SYM). Note that this metric on operators is defined using space-time dependent two-point functions but is itself independent of spacetime. Knowing the metric for arbitrary derivatives allows a reconstruction of the spacetime dependence.

2.1 Oscillator construction

The oscillator representation allows an elegant method of constructing primary fields in the SL(2) sector [26, 27]. By using this representation to find the Clebsch-Gordan coefficients associated with the $SL(2) \times S_n$, we will solve the problem of finding the canonical covariant 2-point functions. It will turn out that in addition to the groups SL(2) and S_n another symmetric group will play an interesting role. It is S_k where k is the number of derivatives required to construct the lowest weight state.

The $SO(4,2)$ conformal algebra is given by

$$[M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad\qquad [M_{ab}, K_c] = \eta_{bc} K_a - \eta_{ac} K_b, \tag{2.3}$$

$$[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} - \eta_{ac} M_{bd} + \eta_{ad} M_{bc} - \eta_{bd} M_{ac}, \tag{2.4}$$

$$[D, P_a] = P_a, \quad [D, K_a] = -K_a, \qquad\qquad [K_a, P_b] = 2\eta_{ab} D - 2M_{ab} \tag{2.5}$$

The SL(2) sector in terms of the conformal generators can be chosen as

$$L_+ = \frac{1}{2}(P_0 + P_3) \qquad L_- = \frac{1}{2}(K_0 - K_3) \qquad L_0 = \frac{1}{2}(D - M_{03}) \tag{2.6}$$

giving

$$[L_-, L_+] = 2\,L_0\,, \qquad [L_0, L_\pm] = \pm L_\pm \tag{2.7}$$

This algebra may be represented using oscillators as

$$L_+ = a^\dagger + a^\dagger a^\dagger a \,, \qquad L_0 = \frac{1}{2} + a^\dagger a \,, \qquad L_- = a \tag{2.8}$$

where $[a, a^\dagger] = 1$. The lowest weight state of the representation V_F is denoted $|0\rangle$ and is annihilated by all the lowering oscillators $L_- = a$. It can straightforwardly be checked that the raising operators L_+ then act on the lowest weight state as

$$(L_+)^k \, |0\rangle = k! \, (a^\dagger)^k \, |0\rangle \qquad \leftrightarrow \qquad \partial^k X \, |0\rangle \tag{2.9}$$

By the operator-state correspondence, the operator on the r.h.s. above acts on the CFT vacuum at the origin in radial quantization to give a state. Hence we have a map from oscillator states used in the representation theory of SL(2) to states in radial quantization. Dual states in the oscillator Hilbert space map to states at the dual vacuum (at infinity) in radial quantization.

$$\langle 0| L_-^k = \langle 0| a^k \qquad \leftrightarrow \qquad \langle 0| \, \partial^k X \tag{2.10}$$

Note that the usual oscillator inner product correctly maps to the inner product given by the Zamolodchikov metric (2.2). The normalization in (2.2) is easily calculated using the SL(2) algebra once we use the fact that L_- is the hermitian conjugate of L_+ in radial quantization.

In a similar way we can represent the tensor product $V_F^{\otimes n}$ by considering n independent oscillators a_i. In this space the action of the diagonal SL(2) is obtained by summing over n:

$$\mathbf{L}_+ = \sum_i (a_i^\dagger + a_i^\dagger a_i^\dagger a_i) \,, \qquad \mathbf{L}_0 = \frac{1}{2} n + \sum_i a_i^\dagger a_i \,, \qquad \mathbf{L}_- = \sum_i a_i \tag{2.11}$$

The relation between the oscillator states and the field states is:

$$\prod_{l=1}^n (a_l^\dagger)^{k_l} \, |0\rangle \qquad \leftrightarrow \qquad \frac{1}{k_1! k_2! \dots k_n!} \partial^{k_1} X \otimes \partial^{k_2} X \otimes \cdots \otimes \partial^{k_n} X \, |0\rangle \tag{2.12}$$

The lowest weights are annihilated by $L_- = \sum_i a_i$. All the lowest weight states at level $L_0 = n+k$ are generated by k-oscillator states obtained as products of $(a_i^\dagger - a_j^\dagger)$ acting on the vacuum. The simplest example is at $n = 2$ where the lowest weight states are all of the form

$$\mathcal{O}_k = (a_1^\dagger - a_2^\dagger)^k |0\rangle \tag{2.13}$$

Expanding out the oscillators and using (2.12) we find the corresponding operators in field space

$$\mathcal{O}_k \sim \sum_{j=0}^k \binom{k}{j}^2 (-1)^{k-j} \, \partial^j X \otimes \partial^{k-j} X \tag{2.14}$$

These are conformal higher spin currents, first constructed in [28].

2.2 Action of S_n on A_a

The lowest weight states are clearly generated by the $n-1$ differences of oscillators $a_i^\dagger - a_j^\dagger$. The following basis will be very useful

$$A_a^\dagger = \sum_{i=1}^n J_a{}^i a_i^\dagger \tag{2.15}$$

where $J_a{}^i$ takes us from the natural representation of S_n on n objects (labelled by the index i) to the $n-1$ dimensional $H = [n-1,1]$ representation for which we will choose the orthonormal basis (labelled by the index a).[1] The matrix J will thus have the following properties

$$D_{ab}^H(\sigma) J_b{}^i = J_a{}^{\sigma(i)} \tag{2.16}$$

$$J_a{}^i (J^\dagger)_b{}^i = \delta_{ab} \tag{2.17}$$

with $D_{ab}^H(\sigma)$ the orthonormal representation. Explicitly we find

$$A_a^\dagger = \frac{1}{\sqrt{a(a+1)}} \left(a_1^\dagger + \ldots a_a^\dagger - a a_{a+1}^\dagger \right) \tag{2.18}$$

The details of the S_n action on $A_a^\dagger$, and its relation to the orthogonal representing matrix of the hook representation $D_{ab}^H(\sigma)$, are given in appendix section B.

2.3 Canonical covariant form using oscillators

The simplest operator in the space $V_F^{\otimes n}$ is given by n scalar fields without derivatives

$$(\mathcal{O}_n)_{j_1 j_2 \cdots j_n}^{i_1 i_2 \cdots i_n} = X_{j_1}^{i_1} \otimes X_{j_2}^{i_2} \otimes \cdots \otimes X_{j_n}^{i_n} \tag{2.19}$$

The scalar fields lie in the adjoint representation of the gauge group $U(N)$. Using multi-indices $I = (i_1, i_2 \cdots i_n)$ and $J = (j_1, j_2 \cdots j_n)$ we can write

$$(\mathcal{O}_n)_J^I = (X \otimes X \otimes \cdots X)_J^I \equiv \mathbf{X}_J^I \tag{2.20}$$

There is a map which takes the states in the oscillator construction of the $V_F^{\otimes n}$ representation of SL(2) to operators or states (by the operator-state correspondence) in the CFT. The vacuum of the oscillator construction maps to $\mathbf{X}_J^I |0\rangle$. Denoting the map from oscillator Hilbert space to the CFT states as ρ_J^I we may write

$$\rho_J^I(|0\rangle) \leftrightarrow \mathbf{X}_J^I |0\rangle \equiv |\mathbf{X}_J^I\rangle \tag{2.21}$$

The vacuum on the right can be viewed as the vacuum in the CFT associated with the origin in radial quantization. The dual oscillator vacuum maps to the dual vacuum at infinity in radial quantization.

$$\rho_J^I(\langle 0|) \leftrightarrow \langle 0| \mathbf{X}^\dagger{}_J^I \equiv \langle \mathbf{X}_J^I| \tag{2.22}$$

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[1] The natural representation of S_n on n objects is a reducible representation and decomposes into the $n-1$ dimensional rep $H = [n-1,1]$ and the 1-dimensional symmetric rep $[n]$. We consider the map from a_i onto the symmetric representation in section 2.4 where we see that they correspond to the application of the SL(2) descendant operator.

The correlation function of these operators is

$$\langle \mathbf{X}^I_J | \mathbf{X}^K_L \rangle = \sum_{\sigma \in S_n} \sigma^I_L (\sigma^{-1})^K_J \tag{2.23}$$

Now the operator corresponding to $a^\dagger_i |0\rangle$ has $n-1$ scalar fields X with one derivative of a scalar field ∂X sitting at position i. Summing over Wick contractions and using the orthogonality in (2.2) gives the following correlation function

$$\rho^I_J \big(\langle 0 | a_i \big) \rho^K_L \big(a^\dagger_j |0\rangle \big) = \sum_{\sigma \in S_n} \delta_{i\sigma(j)} \sigma^I_L (\sigma^{-1})^K_J \tag{2.24}$$

Here one only sums over permutations σ which map position j to i, enforced by the delta function. This in turn leads to the correlation function for k oscillators (k derivatives):

$$\rho^I_J \big(\langle 0 | a_{i_1} a_{i_2} \ldots a_{i_k} | \big) \rho^K_L \big(a^\dagger_{j_1} a^\dagger_{j_2} \ldots a^\dagger_{j_k} |0\rangle \big) = \sum_{\rho \in S_k} \sum_{\sigma \in S_n} \delta_{i_1 \sigma(j_{\rho(1)})} \ldots \delta_{i_k \sigma(j_{\rho(k)})} \sigma^I_L (\sigma^{-1})^K_J \tag{2.25}$$

We wish to rewrite this in terms of the lowest weight states $\rho^K_L \big(A^\dagger_{a_1} A^\dagger_{a_2} \ldots A^\dagger_{a_k} |0\rangle \big)$. The correlation function of these states is then

$$
\begin{aligned}
\rho^I_J \big(\langle 0 | & (A_{a_1} A_{a_2} \ldots A_{a_k}) \, \rho^K_L \big(A^\dagger_{b_1} A^\dagger_{b_2} \ldots A^\dagger_{b_k} |0\rangle \big) \\
&= \sum_{\rho \in S_k} \sum_{\sigma \in S_n} J^{\sigma(j_{\rho(1)})}_{a_1} \ldots J^{\sigma(j_{\rho(k)})}_{a_k} (J^\dagger)^{j_1}_{b_1} \ldots (J^\dagger)^{j_k}_{b_k} \sigma^I_L (\sigma^{-1})^K_J \\
&= \sum_{\rho \in S_k} \sum_{\sigma \in S_n} D^H_{a_1 c_1}(\sigma) \ldots D^H_{a_k c_k}(\sigma) J^{j_{\rho(1)}}_{c_1} \ldots J^{j_{\rho(k)}}_{c_k} (J^\dagger)^{j_1}_{b_1} \ldots (J^\dagger)^{j_k}_{b_k} \sigma^I_L (\sigma^{-1})^K_J \\
&= \sum_{\rho \in S_k} \sum_{\sigma \in S_n} D^H_{a_{\rho(1)} b_1}(\sigma) \ldots D^H_{a_{\rho(k)} b_k}(\sigma) \, \sigma^I_L (\sigma^{-1})^K_J
\end{aligned}
\tag{2.26}
$$

Now notice that $A^\dagger_{a_1} A^\dagger_{a_2} \ldots A^\dagger_{a_k}$ is totally symmetric in the a_i indices and thus lies in the totally symmetric tensor product $\mathrm{Sym}(V^{\otimes k}_H)$. Equivalently this is the projection in

$$V^{\otimes k}_H = \bigoplus_{\Lambda_1, \Lambda_2} V^{(S_n)}_{\Lambda_1} \otimes V^{(S_k)}_{\Lambda_2} \otimes V_{\Lambda_1, \Lambda_2} \tag{2.27}$$

to $\Lambda_2 = [k]$. We can consider the corresponding Clebsch-Gordon coefficients to the irreducible representation Λ_1.

$$C^{\Lambda_1, [k], m_{\Lambda_1}; \tau}_{a_1 \ldots a_k} \tag{2.28}$$

The index τ runs through the multiplicity of the irrep. $\Lambda_1 \otimes [k]$ of $S_n \times S_k$ in $V^{\otimes k}_H$ which is the dimension of $V_{\Lambda, [k]}$. Equivalently, this is the multiplicity of Λ_1 of S_n in $\mathrm{Sym}(V^{\otimes k}_H)$. Formulae for the multiplicities are given in section 3. The Clebsch-Gordan coefficients are explained in appendix section A.

The CG coefficients have the following properties. Firstly

$$C^{\Lambda_1, [k], m_{\Lambda_1}; \tau}_{a_1 \ldots a_k} = C^{\Lambda_1, [k], m_{\Lambda_1}; \tau}_{a_{\rho(1)} \ldots a_{\rho(k)}} \qquad \forall \rho \in S_k \tag{2.29}$$

which simply reflects the fact that the CG coefficient couples to the *symmetric* product $\text{Sym}(V_H^{\otimes k})$. Secondly

$$C_{a_1...a_k}^{\Lambda_1,[k],m_{\Lambda_1};\tau} D_{a_1 b_1}^H(\sigma) \ldots D_{a_k b_k}^H(\sigma) = D_{m_{\Lambda_1} m'_{\Lambda_1}}^{\Lambda_1}(\sigma) C_{b_1...b_k}^{\Lambda_1,[k],m'_{\Lambda_1};\tau} \tag{2.30}$$

which is derived in (A.15). Finally orthogonality:

$$C_{a_1...a_k}^{\Lambda_1,[k],m_{\Lambda_1};\tau} (C^\dagger)_{a_1...a_k}^{\Lambda'_1,[k],m'_{\Lambda'_1};\tau'} = \delta_{\Lambda_1 \Lambda'_1} \delta_{m_{\Lambda_1} m'_{\Lambda_1}} \delta_{\tau\tau'} \tag{2.31}$$

Now define new operators transforming in the irreps. of S_n

$$\left(\mathcal{O}^{n,k;\Lambda_1,m_{\Lambda_1};\tau}\right)_J^I = C_{a_1...a_k}^{\Lambda_1,[k],m_{\Lambda_1};\tau} \rho_J^I \left(A_{a_1}^\dagger \ldots A_{a_k}^\dagger |0\rangle\right) \tag{2.32}$$

Using the properties of the Clebsch-Gordon coefficients we show that the operators $\mathcal{O}^{n,k;\Lambda_1,m_{\Lambda_1};\tau_\Lambda}$ have the following correlation function

$$\left\langle \left(\mathcal{O}^{n,k;\Lambda_1,m_{\Lambda_1};\tau}\right)_J^I \left(\mathcal{O}^{\dagger n',k';\Lambda'_1,m'_{\Lambda'_1};\tau'}\right)_L^K \right\rangle$$

$$= \delta^{nn'} \delta^{kk'} C_{a_1...a_k}^{\Lambda_1,[k],m_{\Lambda_1};\tau} (C^\dagger)_{b_1...b_k}^{\Lambda'_1,[k],m'_{\Lambda'_1};\tau'}$$

$$\times \sum_{\rho \in S_k} \sum_{\sigma \in S_n} D_{a_{\rho(1)} b_1}^H(\sigma) \ldots D_{a_{\rho(k)} b_k}^H(\sigma) \, \sigma_L^I (\sigma^{-1})_J^K$$

$$= \delta^{nn'} \delta^{kk'} k! \, C_{b_1...b_k}^{\Lambda_1,[k],\hat{m}_{\Lambda_1};\tau} (C^\dagger)_{b_1...b_k}^{\Lambda'_1,[k],m'_{\Lambda'_1};\tau'} \sum_{\sigma \in S_n} D_{m_{\Lambda_1} \hat{m}_{\Lambda_1}}^{\Lambda_1}(\sigma) \, \sigma_L^I (\sigma^{-1})_J^K$$

$$= \delta^{nn'} \delta^{kk'} \delta^{\Lambda_1 \Lambda'_1} \delta^{\tau\tau'} k! \sum_{\sigma \in S_n} D_{m_{\Lambda_1} m'_{\Lambda_1}}^{\Lambda_1}(\sigma) \, \sigma_L^I (\sigma^{-1})_J^K \tag{2.33}$$

2.4 Descendants

We have a raising operator $\mathbf{L}_+$ (see equation (2.11)) corresponding to a space-time derivative. Acting on the lowest weight state we obtain the descendant operator

$$\mathcal{O}^{\Lambda=n+k,M_\Lambda;\Lambda_1,m_{\Lambda_1};\tau} = (\mathbf{L}_+)^{M_\Lambda} C_{a_1...a_k}^{\Lambda_1,[k],m_{\Lambda_1};\tau} A_{a_1}^\dagger \ldots A_{a_k}^\dagger |\mathbf{X}_J^I\rangle \tag{2.34}$$

We then find the expected canonical form, using the commutator relations in section 2.1.

$$\left\langle \mathcal{O}^{\Lambda=n+k,M_\Lambda;\Lambda_1,m_{\Lambda_1};\tau} (\mathcal{O}^\dagger)^{\Lambda'=n'+k',M'_{\Lambda'};\Lambda'_1,m'_{\Lambda'_1};\tau'} \right\rangle$$

$$= \delta^{nn'} \delta^{kk'} \delta^{M_\Lambda M'_{\Lambda'}} \delta^{\Lambda_1 \Lambda'_1} \delta^{\tau\tau'} (M_\Lambda!)^2 \, k! \sum_{\sigma \in S_n} D_{m_{\Lambda_1} m'_{\Lambda_1}}^{\Lambda_1}(\sigma) \, \sigma_L^I (\sigma^{-1})_J^K \tag{2.35}$$

3 Multiplicities of $\text{SL}(2) \times S_n$ irreps. in $V_F^{\otimes n}$

In this section we give formulae for the multiplicities of the operators found in (2.32) using characters.

3.1 Multiplicity of $\mathrm{SL}(2)$ irreps.

We begin by considering the multiplicities of $\mathrm{SL}(2)$ irreps in $V_F^{\otimes n}$ which includes a sum over S_n irreps. The states $\partial^l X$ in V_F have weights $L_0 = 1 + l$, with l going up to infinity. They form a lowest weight discrete series irrep $V_1 = V_F$. Similar discrete series irreps exist for any k, i.e. V_k. We wish to find the tensor product decomposition of $V_1^{\otimes n}$ in terms of the irreps. V_k. This can be derived by characters. The character of the irrep. V_k is

$$\chi_k(q) := Tr_{V_k}(q^{L_0}) = q^k \sum_{l=0}^{\infty} q^l = \frac{q^k}{(1-q)} \tag{3.1}$$

For the tensor product $V_1^{\otimes n}$ we get the character

$$
\begin{aligned}
(\chi_1(q))^n &= \frac{q^n}{(1-q)} \frac{1}{(1-q)^{n-1}} \\
&= \frac{q^n}{(1-q)} \sum_{k \geq 0} \frac{(n-2+k)!}{k!(n-2)!} \, q^k \\
&= \sum_{k \geq 0} \chi_{n+k}(q) \; m(k,n)
\end{aligned}
\tag{3.2}
$$

where we have defined

$$m(k,n) = \frac{(n-2+k)!}{k!(n-2)!} \tag{3.3}$$

We thus have the decomposition of $V_F^{\otimes n}$ as:

$$V_F^{\otimes n} = \bigoplus_{k \geq 0} \; m(k,n) \; V_{n+k} \tag{3.4}$$

We have seen in the previous section that the multiplicity $m(k,n)$ is generated by k powers of the oscillators $A_a^\dagger$. They transform in the $H = [n-1,1]$ representation of S_n with dimension $(n-1)$. The k-oscillator states transform in the symmetrised tensor product of the hook representation which does indeed have dimension $m(k,n)$. The multiplicity $m(k,n)$ can be decomposed into irreps of S_n by finding the decomposition of $\mathrm{Sym}(V_H^{\otimes k})$ into irreps of S_n. We can therefore write

$$
\begin{aligned}
V_1^{\otimes n} &= \bigoplus_k V_{\Lambda = n+k} \otimes \mathrm{Sym}(V_H^{\otimes k}) \\
&= \bigoplus_k V_{\Lambda = n+k}^{\mathrm{SL}(2)} \otimes V_{\Lambda_1}^{S_n} \otimes V_{\Lambda,\Lambda_1}^{\mathrm{Com}(\mathrm{SL}(2) \times S_n)}
\end{aligned}
\tag{3.5}
$$

Here the integer k runs from 0 to infinity and $\mathrm{Com}(\mathrm{SL}(2) \times S_n)$ is the commutant of $\mathrm{SL}(2) \times S_n$. Now if we act with a projector $P_{\Lambda_1} = \frac{d_{\Lambda_1}}{n!} \sum_\sigma \chi_{\Lambda_1}(\sigma) \sigma$ on $V_1^{\otimes n}$ we will project out the subspace with a fixed Λ_1. Taking the $\mathrm{SL}(2)$ character in this Λ_1-symmetrised subspace and expanding in terms of the characters of V_p will yield the dimensions of the multiplicity spaces $V_{\Lambda,\Lambda_1}^{\mathrm{Com}(\mathrm{SL}(2) \times S_n)}$. This provides a way to solve the problem of decomposing $\mathrm{Sym}(V_H^{\otimes k})$ into irreps of S_n. We consider this problem in the next section.

3.2 Multiplicities of $\mathrm{SL}(2) \times S_n$ and q-deformed $GL(\infty)$

3.2.1 Examples of symmetric and antisymmetric S_n irreps

As an example of this method, take $\Lambda_1 = [n]$ the symmetric irrep. We want to calculate $tr_{\mathcal{W}} P_{\Lambda_1} q^{L_0}$ where the trace is taken over $\mathcal{W} = V_1^{\otimes n}$. This means calculating q^{L_0} in the symmetrised subspace of $V_1^{\otimes n}$. A basis in the symmetrised subspace of $|m_1, m_2, .., m_n\rangle$ is in $1-1$ correspondence with natural numbers $m_1, m_2, \ldots m_n$ obeying

$$0 \leq m_1 \leq m_2 \leq \cdots m_n \leq \infty \tag{3.6}$$

So the character is

$$
\begin{aligned}
tr_{\mathcal{W}} P_{[n]} q^{L_0} &= q^n \sum_{m_n=0}^{\infty} \sum_{m_{n-1}=0}^{m_n} \cdots \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} q^{m_1+m_2+\cdots+m_n} \\
&= q^n \prod_{i=1}^{n} \frac{1}{(1-q^i)} \\
&= \frac{q^n}{(1-q)} \prod_{i=2}^{n} \frac{1}{(1-q^i)}
\end{aligned} \tag{3.7}
$$

The multiplicity of $V_{\Lambda=n+k}^{\mathrm{SL}(2)} \otimes V_{[n]}^{(S_n)}$ is then the coefficient of q^k in the generating function

$$\prod_{i=2}^{n} \frac{1}{(1-q^i)} \tag{3.8}$$

As an example for $n = 2$, the multiplicity of V_{2+k} is the coefficient of q^k in $\frac{1}{1-q^2}$. This tells us that the symmetric irrep. of S_n only appears for $k = 0, 2, 4, \ldots$ with unit multiplicity.

Similarly, for $R = [1^n]$ we apply the antisymmetric projector to $\mathcal{W}$ we have a basis in correspondence with $(m_1, m_2, \ldots, m_n)$ with $m_1 < m_2 < \cdots < m_n$. So the character is

$$
\begin{aligned}
tr_{\mathcal{W}}(P_{[1^n]} q^{L_0}) &= q^n \sum_{m_n=n-1}^{\infty} \sum_{m_{n-1}=n-2}^{m_n-1} \cdots \sum_{m_2=1}^{m_3-1} \sum_{m_1=0}^{m_2-1} q^{m_1+m_2+\cdots+m_n} \\
&= q^n q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} \frac{1}{1-q^i} \\
&= \frac{q^n}{1-q} q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} \frac{1}{1-q^i}
\end{aligned} \tag{3.9}
$$

So the number of antisymmetric $[1^n]$ irreps. of S_n in the multiplicity space of V_{n+k} is the coefficient of q^k in

$$\frac{q^{\frac{n(n-1)}{2}}}{(1-q^2)\cdots(1-q^n)} \tag{3.10}$$

This multiplicity is zero unless $k \geq \frac{n(n-1)}{2}$. This is as it should be because the antisymmetry condition means that we need $X, \partial X, \ldots \partial^{n-1} X$ which has weight $n + \frac{n(n-1)}{2}$.

3.2.2 The generating function for any $SL(2) \times S_n$ irreps

In fact it turns out we can write down a compact formula for the generating function for the multiplicities of $V_{\Lambda=n+k} \otimes V_{\Lambda_1}$ in $\mathcal{W}$ for any Λ_1. It is given by

$$(1-q)q^{\sum_{i=1} \frac{c_i(c_i-1)}{2}} \prod_b \frac{1}{(1-q^{h_b})} \tag{3.11}$$

The product runs over the boxes of the Young diagram of Λ_1 and h_b is the hook length of the box. c_i is the column length of the i'th column. One can check that this agrees with (3.8) and (3.10) for $R = [n]$ and $R = [1^n]$.

The proof of this generating function, using q-dimensions of $GL(\infty)$, goes as follows.

3.2.3 Proof using q-dimensions of $GL(\infty)$

It is useful to think of the infinite dimensional representation V_F as a limit of finite dimensional representations $V_{\tilde{N}}$ of $GL(\tilde{N})$ for $\tilde{N} \to \infty$. This corresponds to considering fundamental fields of the form $\partial^k X$ for $0 \le k \le \tilde{N} - 1$. States in $V_F^{\otimes n} = V_{\tilde{N}}^{\otimes n}$ of a fixed S_n symmetry given by Young diagram Λ_1, can be labelled by inserting n positive integers from $1 \cdots \tilde{N}$ into the Young diagram, with the numbers strictly decreasing down the columns and weakly increasing along the rows. These are the semi-standard Young tableaux [29]. We will denote by $\vec{m}(R)$ a set of numbers corresponding to a semi-standard Young tableau. This corresponds to operators consisting of the letters $\partial^{m_1-1}X, \ldots, \partial^{m_n-1}X$. So we have

$$tr_{\mathcal{W}} P_{\Lambda_1} q^{L_0} = \sum_{\vec{m}(R)} q^{m_1+m_2+\cdots m_n} \tag{3.12}$$

In a standard basis of $GL(\tilde{N})$, with e_i being the column vector with 1 in the i'th place and 0 elsewhere, the diagonal matrices E_{ii} act as

$$E_{ii} e_m = \delta_{im} e_m \tag{3.13}$$

The weight q^m can be identified with the eigenvalues of $q^{\sum_m m E_{mm}}$. Precisely this generator appears in the computation of the q-dimension of the representation Λ_1 of $U_q(GL(\tilde{N}))$. This sum also appears in studying the decomposition of $V_j^{\otimes n}$ for the SU(2) representation V_j $(2j + 1 = \tilde{N})$ in terms of SU(2) $\times S_n$ [19]. It is known to be

$$q^{n+\sum_i \frac{c_i(c_i-1)}{2}} \prod_{i,j} \frac{(1-q^{\tilde{N}-i+j})}{1-q^{h(i,j)}} \tag{3.14}$$

where i runs along the columns, j runs along the rows of the Young diagram, and $h(i,j)$ is the hook length of the box labelled by (i,j). When $\tilde{N} \to \infty$ in a region of $q < 1$ (where the desired sums converge), the numerator goes to 1 and we get the result

$$tr_{\mathcal{W}} P_{\Lambda_1} q^{L_0} = q^{n+\sum_i \frac{c_i(c_i-1)}{2}} \prod_{i,j} \frac{1}{1-q^{h(i,j)}} \tag{3.15}$$

Factoring out the character of $V_{\Lambda=n+k}$ which is $\frac{q^{n+k}}{(1-q)}$ we get the multiplicity of this representation. This proves the claim that the coefficient of q^k in (3.11) is the multiplicity of the representation $V_{\Lambda=n+k}$ in $\mathcal{W}$.

3.3 The general case using characters of symmetric groups $S_n \times S_k$

As we explained in section 3.1, the oscillator construction of SL(2) representations implies that the multiplicity of the $\mathrm{SL}(2) \times S_n$ representation $V_{\Lambda=n+k} \otimes V_{\Lambda_1}$ in the decomposition of $V_1^{\otimes n}$ is given by the multiplicity of V_{Λ_1} of S_n in $\mathrm{Sym}(V_H^{\otimes k})$. Equivalently this is the multiplicity of the representation $\Lambda_1 \otimes [k]$ of $S_n \times S_k$ in $V_H^{\otimes k}$, where $[k]$ denotes the Young diagram of S_k with a single row of length k which is the symmetric representation. The projectors $P_{\Lambda_1} \otimes P_{[k]}$ can be written down using characters of symmetric groups. Hence we have

$$
\begin{aligned}
d_{\Lambda=n+k,\Lambda_1} &= \mathrm{tr}_{V_H}(P_{\Lambda_1} \otimes P_{[k]}) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\Lambda_1}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \chi_{[k]}(\tau) \prod_i (\mathrm{tr}_{V_H}(\sigma^i))^{c_i(\tau)}
\end{aligned}
\tag{3.16}
$$

$c_i(\tau)$ is the number of cycles in τ of length i. See appendix section A.1 for further details.

We can check that this multiplicity gives the correct $m(k,n)$

$$
\begin{aligned}
m(k,n) &= \sum_{\Lambda_1(S_n)} d_{\Lambda_1} d_{\Lambda=n+k,\Lambda_1} \\
&= \sum_{\sigma \in S_n} \sum_{\Lambda_1(S_n)} \frac{1}{n!} d_{\Lambda_1} \chi_{\Lambda_1}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \prod_i (\mathrm{tr}_{V_H}(\sigma^i))^{c_i(\tau)} \\
&= \frac{1}{k!} \sum_{\tau \in S_k} (\mathrm{tr}_{V_H}(\mathrm{id}))^{\sum_i c_i(\tau)} \\
&= \dim_{n-1}[k] \\
&= \frac{(n-2+k)!}{k!(n-2)!}
\end{aligned}
\tag{3.17}
$$

We have identified $m(k,n)$ with the dimension of the totally symmetric $GL(n-1)$ representation $[k]$.

4 General G, V

We now consider how the derivation of the canonical covariant form, such as (2.33) in the SL(2) case, generalises to the situation of any global symmetry group G. The Lie algebra generators act on n-fold tensor products of representations $V_1 \otimes V_2 \cdots \otimes V_n$ as

$$
\Delta_n(J_a) = J_a \otimes 1 \otimes \cdots \otimes 1 \ + \ 1 \otimes J_a \otimes \cdots \otimes 1 \ + \ \ldots \ + \ 1 \otimes \cdots \otimes J_a
\tag{4.1}
$$

In particular we will be interested in the n-fold tensor product of the representation V_F corresponding to the fundamental fields in (a sector of) the theory. Note that the action of G commutes with the symmetric group action permuting the n factors in the tensor product. For any $\sigma \in S_n$ we have in $End(V_F^{\otimes n})$

$$
\sigma \, \Delta_n(J_a) = \Delta_n(J_a) \, \sigma
\tag{4.2}
$$

The signs that arise in the super-algebra case will be discussed in section 4.1.1, where we will show that the key result generalizes.

We will now organise the states in $V_F^{\otimes n}$ according to the representations of the product group $G \times S_n$ acting on this space

$$(V_F^G)^{\otimes n} = \oplus_{\Lambda, \Lambda_1} V_\Lambda^G \otimes V_{\Lambda_1}^{S_n} \otimes V_{\Lambda, \Lambda_1}^{\text{Com}(G \times S_n)} \tag{4.3}$$

The n-fold tensor product has states

$$|m_1, m_2, \ldots, m_n\rangle \tag{4.4}$$

where m runs over the states in the fundamental representation (which can be infinite dimensional as in the SL(2) case). We can choose an orthonormal basis

$$\langle m_1, m_2, \ldots m_n | m_1', m_2', \ldots, m_n'\rangle = \delta_{m_1 m_1'} \cdots \delta_{m_n m_n'} \tag{4.5}$$

We can decompose the n-fold tensor product into irreps of the global symmetry as follows

$$|m_1, \ldots, m_n\rangle = \sum_{\Lambda, M_\Lambda, i} C_{\vec{m}}^{\Lambda, M_\Lambda, i} |\Lambda, M_\Lambda, i\rangle \tag{4.6}$$

M_Λ is the state within V_Λ and i is a multiplicity index for V_Λ. In fact $V^{\otimes n}$ has an action of $G \times S_n$ so we can decompose the above multiplicity index i into $(\Lambda_1, m_{\Lambda_1}, \tau)$ where Λ_1 labels an S_n irrep, m_{Λ_1} runs over the states in the irrep Λ_1 of S_n and τ labels the multiplicity of $V_\Lambda \otimes V_{\Lambda_1}$. We define the Clebsch for this decomposition

$$C_{\vec{m}}^{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} = \langle \Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau | \vec{m} \rangle \tag{4.7}$$

The Clebsch are invertible

$$|\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau\rangle = \sum_{\vec{m}} C_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^{\vec{m}} |\vec{m}\rangle \tag{4.8}$$

so that

$$C_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^{\vec{m}} = \langle \vec{m} | \Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau \rangle \tag{4.9}$$

Using the hermiticity of the inner product

$$C_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^{\vec{m}} = \left(C_{\vec{m}}^{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} \right)^* \tag{4.10}$$

We can always choose an orthonormal basis

$$\langle \Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau | \Lambda', M_{\Lambda'}', \Lambda_1', m_{\Lambda_1'}', \tau' \rangle = \delta_{\Lambda \Lambda'} \delta_{M_\Lambda M_{\Lambda'}'} \delta_{\Lambda_1 \Lambda_1'} \delta_{m_{\Lambda_1} m_{\Lambda_1'}'} \delta_{\tau \tau'} \tag{4.11}$$

which leads to orthogonality of the Clebsch

$$\sum_{\vec{m}} \left(C_{\vec{m}}^{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} \right)^* C_{\vec{m}}^{\Lambda', M_{\Lambda'}', \Lambda_1', m_{\Lambda_1'}', \tau'} = \delta_{\Lambda \Lambda'} \delta_{M_\Lambda M_{\Lambda'}'} \delta_{\Lambda_1 \Lambda_1'} \delta_{m_{\Lambda_1} m_{\Lambda_1'}'} \delta_{\tau \tau'} \tag{4.12}$$

4.1 Correlators in free field theory

Corresponding to the orthogonal states in V_F we have fields W_m which diagonalize the Zamolodchikov metric

$$\langle W_m W_{m'} \rangle = \delta_{mm'} \tag{4.13}$$

For the states in $V_F^{\otimes n}$ take operators $\mathcal{O}_{m_1,m_2,\ldots m_n} \equiv \mathcal{O}_{\vec{m}} = W_{m_1} \otimes W_{m_2} \cdots W_{m_n}$. In the case of SL(2) this is

$$\frac{1}{m_1! m_2! \cdots m_n!} \partial^{m_1} X \otimes \partial^{m_2} X \cdots \otimes \partial^{m_n} X \tag{4.14}$$

The 2-point function can be written as

$$\langle (\, \mathcal{O}_{m_1,m_2,\ldots m_n}^\dagger \,)_J^I, (\, \mathcal{O}_{m_1',m_2'\cdots m_n'} \,)_L^K \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^n \delta_{m_i m_{\sigma(i)}'} \; (\sigma)_J^K (\sigma^{-1})_L^I \tag{4.15}$$

where the sum over σ runs over Wick contractions and we have used the orthogonality (4.13). Define operators in correspondence with the orthonormal $G \times S_n$ basis of the $V^{\otimes n}$:

$$\mathcal{O}_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} = \sum_{\vec{m}} C_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^{\vec{m}} \mathcal{O}_{m_1,\ldots,m_n} \tag{4.16}$$

The correlator of the gauge-covariant operators is

$$\langle (\mathcal{O}_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^\dagger)_J^I \, (\mathcal{O}_{\Lambda', M_{\Lambda'}', \Lambda_1', m_{\Lambda_1'}', \tau'})_L^K \rangle$$

$$= \sum_{\vec{m}, \vec{m}'} \left(C_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^{m_1 \cdots m_n} \right)^* C_{\Lambda', M_{\Lambda'}', \Lambda_1', m_{\Lambda_1'}', \tau'}^{m_1' \cdots m_n'} \sum_{\sigma \in S_n} \prod_{i=1}^n \delta_{m_i m_{\sigma(i)}'} \; (\sigma)_J^K (\sigma^{-1})_L^I \tag{4.17}$$

Solve the delta function

$$\langle (\mathcal{O}_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^\dagger)_J^I \, (\mathcal{O}_{\Lambda', M_{\Lambda'}', \Lambda_1', m_{\Lambda_1'}', \tau'})_L^K \rangle$$

$$= \sum_{\sigma \in S_n} \sum_{\vec{m}} \left(C_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^{m_1 \cdots m_n} \right)^* C_{\Lambda', M_{\Lambda'}', \Lambda_1', m_{\Lambda_1'}', \tau'}^{m_{\sigma^{-1}(1)} \cdots m_{\sigma^{-1}(n)}} \; (\sigma)_J^K (\sigma^{-1})_L^I \tag{4.18}$$

We can simplify the σ^{-1} action on $V_F^{\otimes n}$ in the second Clebsch because we know it transforms under the S_n representation Λ_1': the action is just the matrix representation of Λ_1'.

$$C_{\Lambda', M_{\Lambda'}', \Lambda_1', m_{\Lambda_1'}', \tau'}^{m_{\sigma^{-1}(1)} \cdots m_{\sigma^{-1}(n)}} = D_{m_{\Lambda_1'}' \hat{m}_{\Lambda_1'}}^{\Lambda_1'} (\sigma^{-1}) \; C_{\Lambda', M_{\Lambda'}', \Lambda_1', \hat{m}_{\Lambda_1'}, \tau'}^{m_1 \cdots m_n} \tag{4.19}$$

Use this and the orthogonality of the Clebsch from equation (4.12) to get

$$\langle (\mathcal{O}_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^\dagger)_J^I (\mathcal{O}_{\Lambda', M_{\Lambda'}', \Lambda_1', m_{\Lambda_1'}', \tau'})_L^K \rangle$$

$$= \delta_{\Lambda \Lambda'} \delta_{M_\Lambda M_{\Lambda'}'} \delta_{\Lambda_1 \Lambda_1'} \delta_{\tau \tau'} \sum_{\sigma \in S_n} D_{m_{\Lambda_1} m_{\Lambda_1'}'}^{\Lambda_1} (\sigma) \; (\sigma)_J^K (\sigma^{-1})_L^I \tag{4.20}$$

4.1.1 Signs in the super-algebra case

In the case where the generators of G include fermionic ones, there is a small modification of the above proof. Fermionic generators Q pick up signs when taken past fermionic fields

$$Q(\psi_1\psi_2) = (Q\psi_1)\psi_2 - \psi_1(Q\psi_2) \qquad (4.21)$$

In this case the action of permutations is defined with a sign for each fermion swap. For the transposition of $\psi_1\psi_2$ we define

$$s(\psi_1\psi_2) = -\psi_2\psi_1 \qquad (4.22)$$

It is easy to check that $Qs = sQ$. This is the key point. If we define the action of permutations to pick up a sign for every swap of fermions, we have an action of the permutation group which commutes with the super-algebra. Hence Clebsch-Gordan coefficients for $G \times S_n$ are well-defined, and we can define operators according to (4.16). The two-point function in this case picks up a sign $(-1)^{\epsilon(\vec{m},\sigma)}$ on the r.h.s. of (4.17). This sign carries into (4.18). Because the correct action of the permutations involves this same sign factor, equation (4.19) has the sign $(-1)^{\epsilon(\vec{m},\sigma)}$ on the left. The final result (4.20) remains unchanged.

4.2 Completeness

The completeness of these operators in $V_F^{\otimes n}$ follows from the invertibility of the Clebschs; this means that any state in $V_F^{\otimes n}$ can be written as a linear combination of states in $V_\Lambda \otimes V_{\Lambda_1}$.

$$\mathcal{O}_{m_1,\ldots m_n} = \sum_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} C_{\vec{m}}^{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} \mathcal{O}_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} \qquad (4.23)$$

If we reintroduce the gauge indices and trace our covariant operators to get gauge-invariant operators, then we must also take into account finite N constraints. This is studied in section 6.

5 Examples and applications

The above formalism can be applied to any global symmetry G of a theory, or to a subgroup of the global symmetry acting on a sector of the fields. In this section we will consider a number of examples to which we can apply this formalism. Examples relating to $\mathcal{N}=4$ SYM are the SL(2) sector described above and the U(M) (BPS) sector considered in [15]. We will review both of these sectors from the new perspective of the previous section. Free conformal field theories also possess an enlarged symmetry group known as higher spin symmetry and we will also see in this section how our construction naturally assembles into representations of this.

Another sector we will consider corresponds to the 6 hermitian scalars of $\mathcal{N} = 4$ SYM and the SO(6) subgroup of the global symmetry acting on them. The problem of diagonalising the gauge invariant operators amounts to first finding a manageable form of the Clebsch-Gordan coefficients for the SO(6) $\times$ S_n decomposition of the n-fold tensor

product of the fundamental of SO(6). We will not solve this problem explicitly but will note some facts about the relevant multiplicities in section 5.4. A more explicit description of the Clebsch-Gordans analogous to what we gave for U(M) in section 5.2 would be desirable.

The sector of six scalars can also be described in terms of the 3 complex scalars and their conjugates. In this case, it is natural to use the U(3) subgroup of SO(6). The case of purely holomorphic operators was solved in [15] and is reviewed in 5.2. It should be possible to include the anti-holomorphic operators by using U(3) along with the Brauer algebra $B_N(m,n)$ used in [14] for the case of a single complex scalar.

Finally we will look at the formalism applied to product groups. To handle a class of derivatives of the holomorphic scalars we can use the product group technology applied to SL(2) × U(3). A simple sector including one chiral fermion, 3 holomorphic scalars is controlled by SL(2) × $U(3|1)$. For the six hermitian scalars and a class of their derivatives we can use SL(2)×SO(6). For more general derivatives we can use SO(4, 2)×SO(6). For the sixteenth BPS sector we would use SU(3|2, 1) [34].For the complete set of fields of $\mathcal{N} = 4$ SYM we can use the full symmetry SU(2, 2|4). Calculating the Clebsch-Gordan coefficients for the $V_F^{\otimes n}$ will allow, following the derivation of (4.20) from (4.12), to get the canonical covariant form which in turn leads to diagonal gauge invariant operators using section 6. The relevant Clebsch-multiplicities are known for $n = 2$ [26]. Finding the multiplicities and the Clebsch-coefficients in terms of symmetric groups, as we do below for U(M) and SL(2), is the next step in the solution of the free field diagonalisation problem for $\mathcal{N} = 4$ SYM.

5.1 SL(2) and $\mathbf{Sym}(V_H^{\otimes k})$

Here we illustrate our scheme with the construction of the Clebsch $C_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau}^{\vec{m}}$ for the specific example of $G = $ SL(2), which we considered in section 2.

In the Fock space of oscillators we have

$$|m_1, \ldots, m_n\rangle \sim (a_1^\dagger)^{m_1} \ldots (a_n^\dagger)^{m_n}|0\rangle \tag{5.1}$$

whereas the oscillators A_a give the decomposition in terms of SL(2) irreps $\Lambda = n + k$, following equation (4.6)

$$|\Lambda, M; (a_1, \ldots a_k)\rangle = C_{\vec{m}}^{\Lambda, M, (a_1 \ldots a_k)} |m_1, \ldots, m_n\rangle = A_{(a_1}^\dagger \ldots A_{a_k)}^\dagger (\mathbf{L}_+)^M |0\rangle \tag{5.2}$$

so the label i in equation (4.6) is given by the indices $(a_1, \ldots a_k)$. The Clebsch-Gordon coefficients of equation (4.6) can now be read off.

The next step is to decompose the label i into irreps of S_n. The indices a_i carry the $n-1$ dimensional hook representation, $H = [n - 1, 1]$ of S_n. Therefore the label $i = (a_1 \ldots a_k)$ carries the reducible representation $\text{Sym}V_H^{\otimes k}$. This decomposes into the irreducible representations Λ_1 of S_n with multiplicity τ via the Clebsch Gordon coefficient $C_{(a_1 \ldots a_k)}^{\Lambda_1, m_{\Lambda_1}, \tau}$

$$|(a_1 \ldots a_k)\rangle = C_{\Lambda_1, m_{\Lambda_1}, \tau}^{(a_1 \ldots a_k)} |\Lambda_1, m_{\Lambda_1}, \tau\rangle \tag{5.3}$$

and the inverse transformation is given by $C_{\Lambda_1, m_{\Lambda_1}, \tau}^{(a_1 \ldots a_k)}$ as

$$|\Lambda_1, m_{\Lambda_1}, \tau\rangle = C_{(a_1 \ldots a_k)}^{\Lambda_1, m_{\Lambda_1}, \tau} |(a_1 \ldots a_k)\rangle \tag{5.4}$$

Putting all this together we get

$$
\begin{aligned}
|\Lambda, M, \Lambda_1, m_{\Lambda_1}, \tau\rangle &= C_{(a_1 \ldots a_k)}^{\Lambda_1, m_{\Lambda_1}, \tau} |\Lambda, M; (a_1, \ldots a_n)\rangle \\
&= C_{(a_1 \ldots a_k)}^{\Lambda_1, m_{\Lambda_1}, \tau} C_{\vec{m}}^{\Lambda, M, (a_1 \ldots a_k)} |m_1, \ldots, m_n\rangle
\end{aligned}
\tag{5.5}
$$

The Clebsch of equation (4.8) is given by

$$
C_{\vec{m}}^{\Lambda, M, \Lambda_1, m_{\Lambda_1}, \tau} = C_{(a_1 \ldots a_k)}^{\Lambda_1, m_{\Lambda_1}, \tau} C_{\vec{m}}^{\Lambda, M, (a_1 \ldots a_k)}
\tag{5.6}
$$

5.2 U(M) revisited

This formalism also applies to the U(M) case studied in [15]. In that case the starting point is

$$
|m_1, \ldots, m_n\rangle \sim X_{m_1} \otimes \cdots \otimes X_{m_n} \qquad m_i = 1 \ldots M
\tag{5.7}
$$

which is the tensor product of n fundamentals of U(M), $V_M^{\otimes n}$. Then in (4.6) $C_{\vec{m}}^{\Lambda, M, i}$ is the U(M) Clebsch Gordon coefficient decomposing $V_M^{\otimes n}$ into irreps Λ with multiplicity i. In this case Schur Weyl duality tells us that i also carries the fundamental of the representation Λ of S_n. Thus $\Lambda_1 = \Lambda$ and there is no τ multiplicity.

To be explicit, to get the operators of [15]

$$
C_{\Lambda, M, \Lambda_1 = \Lambda, m_{\Lambda_1} = i}^{\vec{m}} = \frac{1}{n!} \sum_{\sigma \in S_n} B_{j\beta} D_{ij}^{\Lambda}(\sigma) \prod_{k=1}^{n} \delta_{m_k p_{\sigma^{-1}(k)}}
\tag{5.8}
$$

Here $M = (\mu, \beta)$. μ labels the number of different flavour fields in the operator (μ_1 X's, μ_2 Y's, etc.), while β runs over the number times the trivial representation of $H_\mu = S_{\mu_1} \times \cdots S_{\mu_M}$ is contained in S_n. $B_{j\beta}$ is a branching coefficient for the change of basis for the subspace of the irrep. Λ invariant under H_μ. Canonically we choose $p_1, \ldots p_{\mu_1} = 1$, $p_{\mu_1+1}, \ldots p_{\mu_1+\mu_2} = 2, \ldots$. With this choice we recover the covariant operators in [15]

$$
\mathcal{O}_{i\beta}^{\Lambda\mu} = \sum_{\vec{m}} C_{\Lambda, \mu, \beta, i}^{\vec{m}} X_{m_1} \ldots X_{m_n} = \frac{1}{n!} \sum_{\sigma \in S_n} B_{j\beta} D_{ij}^{\Lambda}(\sigma) \, \sigma \mathbf{X}^\mu \sigma^{-1}
\tag{5.9}
$$

We then find the orthogonality we expect, up to a normalisation factor

$$
\sum_{\vec{m}} C_{\Lambda, \mu, \beta, i}^{\vec{m}} C_{\Lambda', \mu', \beta', i'}^{\vec{m}} = \delta_{\Lambda\Lambda'} \delta_{\mu\mu'} \delta_{\beta\beta'} \delta_{ii'} \frac{|H_\mu|}{n! d_\Lambda}
\tag{5.10}
$$

5.3 The higher spin group

The free theory of $\mathcal{N}=4$ SYM is invariant under an infinite dimensional group $HS(2, 2|4)$ known as the higher spin group. In the interacting theory this is broken to the superconformal group $SU(2, 2|4)$ but it can nevertheless be useful for some applications (eg possible relations via AdS/CFT to a possible 'tensionless limit' of string theory) to consider this enlarged group. When restricted to the SL(2) sector the higher spin group is known as $HS(1, 1)$. Operators form lowest weight representations of $HS(1, 1)$ (which further decompose into an infinite number of lowest weight representations of SL(2).) The lowest

weight states of these representations were decscribed in [27]. In terms of the oscillators introduced in section 2.1, the higher spin algebra is spanned by the generators

$$J_{p,q} = \sum_i (a_i^\dagger)^p (a_i)^q \tag{5.11}$$

which clearly contains the SL(2) algebra (2.11).

If we consider truncating the fundamental fields so that we only consider states $|m\rangle = (a^\dagger)^m |0\rangle$ for $m \leq M - 1$ then the higher spin group truncates to U(M). Therefore the covariant canonical form corresponding to the higher spin group is simply the $M \to \infty$ limit of that in the previous subsection.

Therefore the results of the previous subsection generalise naturally to the higher spin case. Irreducible representations of the higher spin group are specified by Young tableaux, Λ_1, (as observed in [27]). We have

$$V_F^{\otimes n} = \bigoplus_{\Lambda_1 \vdash n} V_{\Lambda_1}^{HS} \otimes V_{\Lambda_1}^{S_n} \tag{5.12}$$

$$= \bigoplus_{\Lambda_1, \Lambda} V_\Lambda^{\text{SL}(2)} \otimes V_{\Lambda, \Lambda_1}^{\text{Com}(\text{SL}(2) \times S_n)} \otimes V_{\Lambda_1}^{(S_n)} \tag{5.13}$$

The first line is the standard Schur-Weyl duality for U(M) in the limit $M \to \infty$. Each higher spin representation, Λ_1, then decomposes further into an SL(2) irrep Λ and the commutant. The Clebsch gving the first line is given by (5.8). The canonical covariant operators which form irreps of the higher spin group are given by (5.9) with X_m replaced by $(1/m!)\partial^m X$.

5.4 The SO(6) sector

We have 6 hermitian scalar matrices in $\mathcal{N} = 4$ SYM, transforming in the fundamental of SO(6). We know from the general discussion in section 4 that the SO(6) covariant diagonalisation of free field correlators will be solved once we have solved the Clebsch-Gordan problem for SO(6) $\times S_n$ in $V^{\otimes n}$. Here V is the fundamental of SO(6).

$$\begin{aligned} V^{\otimes n} &= \bigoplus_{\Lambda_1} V_{\Lambda_1}^{(S_n)} \otimes V_{\Lambda_1}^{GL(6)} \\ &= \bigoplus_{\Lambda_1, \Lambda_2} V_{\pi(\Lambda_2)}^{(\text{SO}(6))} \otimes V_{\Lambda_1, \Lambda_2} \otimes V_{\Lambda_1}^{(S_n)} \end{aligned} \tag{5.14}$$

We first decompose the n-fold tensor space according to the S_n symmetry. The Schur-Weyl dual of S_n is $GL(6)$ hence the decomposition in the first line. In the second line, we decompose the $GL(6)$ representations to SO(6) representations. The dimension of the multiplicity space V_{Λ_1, Λ_2} is given by

$$Dim V_{\Lambda_1, \Lambda_2} = \sum_\delta g(\Lambda_2, 2\delta; \Lambda_1) \tag{5.15}$$

Λ_1 is a Young diagram with n boxes, 2δ is a partition with even parts, i.e. Young diagram with even row lengths. The sum above includes a sum over $k \geq 0$, where $2k$ is the number of boxes in 2δ and $n - 2k$ is the number of boxes in Λ_2.

The representations of $GL(6)$ are labelled by Young diagrams with row lengths $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_6 \geq 0$. The representations of SO(6) are labelled by $\lambda_1 \geq \lambda_2 \geq |\lambda_3| \geq 0$. The last label λ_3 can be positive or negative. For $\lambda_3 = 0$ the irreps are constructed by symmetrising according to the Young diagram and projecting out traces. When $|\lambda_3| > 0$ the corresponding operation of Young-symmetrising and removing traces leaves us with a reducible representation, which is a direct sum of irreps. $(\lambda_1, \lambda_2, \lambda_3) \oplus (\lambda_1, \lambda_2, -\lambda_3)$. The operation π which appears in (5.14), when it acts on any $GL(6)$ Young diagram Λ_1 gives either zero or a Young diagram obeying the SO(6) constraints. It is defined in terms of an operation on Young diagrams in [31].

We have arrived above at the $SO(6) \times S_n$ decomposition by first decomposing into $GL(6) \times S_n$, then reducing the $GL(6)$ to SO(6). We can equally start by decomposing in terms of $SO(6) \times E_6(n)$ where $E_6(n)$ is the commutant of SO(6) in $V^{\otimes n}$ described for example in [32]. A subsequent decomposition of $E_6(n)$ to S_n should yield the same result as (5.14). This follows from general theorems on double commutants which assert that if A is a subalgebra of B, and $End(B) \subset End(A)$ are their commutants in some vector space, then the reduction multiplicities for irreps of $B \to A$ coincide with those of $End(A) \to End(B)$ (see [17]). In this case the reduction multiplicities of $GL(6) \to SO(6)$ coincide with those of $E_6(n) \to S_n$.

5.5 The $SO(4,2)$ sector

In considering the sector of a scalar field X with all four derivatives acting on it, we can use the $SO(4,2)$ symmetry. Generalizing the linear combinations $A_a^\dagger$ of oscillators which generate the lowest weights in the SL(2) sector, we now have $A_{a\lambda}^\dagger$ where λ is an index in the fundamental of $SO(4) \subset SO(4,2)$ and as before a is in the hook representation $V_H = [n-1,1]$ of S_n. Lowest weights annihilated by K_λ, with k derivatives acting on n-field composites can be constructed from oscillators of the form

$$A_{a_1\lambda_1}^\dagger A_{a_2\lambda_2}^\dagger \cdots A_{a_k\lambda_k}^\dagger |0\rangle \tag{5.16}$$

The simplest class of such LWS are those in which the indices $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ are taken to be a symmetric traceless SO(4) tensor corresponding to the SO(4) Young diagram $[k]$. These states satisfy a type of extremality condition $L_0 = n + k$. More generally we will have states of the form (5.16) which involve contractions of the λ_i. In these cases we have to mod out by the equations of motion, which leads to a projection of the $Sym(V_H \otimes V_H)$ representation of $A_{a_1\lambda}^\dagger A_{a_2\lambda}^\dagger$ to the S_n representation $[n-2,2]$. This has dimension $\frac{n(n-3)}{2}$ which is the number obtained by subtraction of n, for the equations of motion, from the dimension $\frac{n(n-1)}{2}$ of $Sym(V_H \otimes V_H)$. Work on a complete solution of the diagonalisation in this sector, using the above facts to give a symmetric group description of the $SO(4,2) \times S_n$ Clebsch-Gordans, is in progress. It is clear that the symmetric SO(4) operators involving the contractions will have $L_0 > n + k$. The 'extremal' operators mentioned above will be useful in the comparison to excitations of half-BPS giants in section 7.

5.6 Fields carrying reps of product groups

Suppose the global symmetry group has the form $G_1 \times G_2$. We consider a field $\Psi_{k,m}$ where k is an index transforming under irrep V_1 of G_1 and m transforms under irrep V_2 of G_2. Consider the covariant operator

$$(\mathcal{O}_{k_1,m_1;k_2,m_2;\cdots;k_n,m_n})_J^I \equiv (\Psi_{k_1,m_1})_{j_1}^{i_1}(\Psi_{k_2,m_2})_{j_2}^{i_2}\cdots(\Psi_{k_n,m_n})_{j_n}^{i_n} \tag{5.17}$$

Fields with n factors transform under the irrep $(V_1 \otimes V_2)^{\otimes n}$. With $\sigma \in S_n$ acting simultaneously on V_1 and V_2, the commutant of $G_1 \times G_2$ contains S_n. We can consider the group $G_1 \times G_2 \times S_n$ acting on the n-field composites. Correspondingly there is a decomposition of the n-fold tensor product into irreps. of $G_1 \times G_2 \times S_n$. The irreps are related to the product states as

$$|\Lambda_1, M_{\Lambda_1}, \Lambda_2, M_{\Lambda_2}, \Lambda_3, m_{\Lambda_3}, \tau\rangle = C^{\vec{k},\vec{m}}_{\Lambda_1,M_{\Lambda_1},\Lambda_2,M_{\Lambda_2},\Lambda_3,m_{\Lambda_3},\tau}\left|\vec{k},\vec{m}\right\rangle \tag{5.18}$$

Λ_1 is an irrep of G_1, Λ_2 of G_2 and Λ_3 of S_n. Conversely

$$\left|\vec{k},\vec{m}\right\rangle = C^{\Lambda_1,M_{\Lambda_1},\Lambda_2,M_{\Lambda_2},\Lambda_3,m_{\Lambda_3},\tau}_{\vec{k},\vec{m}}|\Lambda_1, M_{\Lambda_1}, \Lambda_2, M_{\Lambda_2}, \Lambda_3, m_{\Lambda_3}, \tau\rangle \tag{5.19}$$

In terms of vector spaces this decomposition is

$$(V_1 \otimes V_2)^{\otimes n} = \bigoplus_{\Lambda_1,\Lambda_2,\Lambda_3} V_{\Lambda_1}^{G_1} \otimes V_{\Lambda_2}^{G_2} \otimes V_{\Lambda_3}^{S_n} \otimes V_{\Lambda_1,\Lambda_2,\Lambda_3}^{\mathrm{Com}(G_1\times G_2\times S_n)} \tag{5.20}$$

τ over the multiplicity space $V_{\Lambda_1,\Lambda_2,\Lambda_3}^{\mathrm{Com}(G_1\times G_2\times S_n)}$.

The orthogonality of Clebsch-Gordan coefficients is the same as before and if we define operators

$$(\mathcal{O}_{\Lambda_1,M_{\Lambda_1},\Lambda_2,M_{\Lambda_2},\Lambda_3,m_{\Lambda_3},\tau})_J^I = \sum_{\vec{k},\vec{m}} C^{\vec{k},\vec{m}}_{\Lambda_1,M_{\Lambda_1},\Lambda_2,M_{\Lambda_2},\Lambda_3,m_{\Lambda_3},\tau}(\mathcal{O}_{\vec{k},\vec{m}})_J^I \tag{5.21}$$

then the covariant two-point function is diagonal in the $(\Lambda_1, M_{\Lambda_1}, \Lambda_2, M_{\Lambda_2}, \Lambda_3, \tau)$ indices, exactly analogous to the single group case in equation (4.20).

5.6.1 Product Clebsch in terms of single group Clebschs

Another way that we could organise $(V_1 \otimes V_2)^{\otimes n}$, in contrast to the $G_1 \times G_2 \times S_n$ decomposition in (5.20), is in terms of the separate groups

$$(V_1 \otimes V_2)^{\otimes n} = V_1^{\otimes n} \otimes V_2^{\otimes n}$$
$$= \left(\bigoplus_{\Lambda_1,\Lambda_4} V_{\Lambda_1}^{G_1} \otimes V_{\Lambda_4}^{S_n} \otimes V_{\Lambda_1,\Lambda_4}^{\mathrm{Com}(G_1\times S_n)}\right) \otimes \left(\bigoplus_{\Lambda_2,\Lambda_5} V_{\Lambda_2}^{G_2} \otimes V_{\Lambda_5}^{S_n} \otimes V_{\Lambda_2,\Lambda_5}^{\mathrm{Com}(G_2\times S_n)}\right)$$

We use the Clebsch $C^{\Lambda_1,M_{\Lambda_1},\Lambda_4,m_{\Lambda_4},\tau_1}_{\vec{k}}$ for G_1 and $C^{\Lambda_2,M_{\Lambda_2},\Lambda_5,m_{\Lambda_5},\tau_2}_{\vec{m}}$ for G_2. Given the simultaneous action of S_n on $(V_1 \otimes V_2)^{\otimes n}$, to connect this decomposition with that in (5.20)

we tensor together the two S_n irreps $V_{\Lambda_4}^{S_n}$ and $V_{\Lambda_5}^{S_n}$ to get the irrep of the simultaneous S_n action $V_{\Lambda_3}^{S_n}$

$$V_{\Lambda_4}^{S_n} \otimes V_{\Lambda_5}^{S_n} = \bigoplus_{\Lambda_3} V_{\Lambda_3}^{S_n} \, C(\Lambda_4, \Lambda_5; \Lambda_3) \tag{5.22}$$

$C(\Lambda_4, \Lambda_5; \Lambda_3)$ counts the number of times $V_{\Lambda_3}^{S_n}$ appears in the S_n tensor product $V_{\Lambda_4}^{S_n} \otimes V_{\Lambda_5}^{S_n}$. This construction shows us how to write down the relation between the $G_1 \times G_2 \times S_n$ Clebsch and the $(G_1 \times S_n) \times (G_2 \times S_n)$ Clebschs

$$C_{\vec{k},\vec{m}}^{\Lambda_1,M_{\Lambda_1},\Lambda_2,M_{\Lambda_2},\Lambda_3,m_{\Lambda_3},\tau} = C_{\vec{k}}^{\Lambda_1,M_{\Lambda_1},\Lambda_4,m_{\Lambda_4},\tau_1} \, C_{\vec{m}}^{\Lambda_2,M_{\Lambda_2},\Lambda_5,m_{\Lambda_5},\tau_2} \, C_{m_{\Lambda_3} \ m_{\Lambda_4} \ m_{\Lambda_5}}^{\tau_3 \Lambda_3 \ \ \Lambda_4 \ \ \Lambda_5} \tag{5.23}$$

The S_n Clebsch-Gordan coefficient $C_{m_{\Lambda_3} \ m_{\Lambda_4} \ m_{\Lambda_5}}^{\tau_3 \Lambda_3 \ \ \Lambda_4 \ \ \Lambda_5}$ gives the change of basis for the decomposition in (5.22); it maps the states of the reps in $V_{\Lambda_4}^{S_n} \otimes V_{\Lambda_5}^{S_n}$ to those in $V_{\Lambda_3}^{S_n}$. τ_3 labels the $C(\Lambda_4, \Lambda_5; \Lambda_3)$ degeneracy. The τ which labels the product group commutant $V_{\Lambda_1,\Lambda_2,\Lambda_3}^{\mathrm{Com}(G_1 \times G_2 \times S_n)}$ is now a combination of the separate group multiplicities and the S_n tensor label τ_3: $\tau = (\tau_1, \tau_2, \tau_3)$.

$$V_{\Lambda_1,\Lambda_2,\Lambda_3}^{\mathrm{Com}(G_1 \times G_2 \times S_n)} = \bigoplus_{\Lambda_4,\Lambda_5} V_{\Lambda_1,\Lambda_4}^{\mathrm{Com}(G_1 \times S_n)} \otimes V_{\Lambda_2,\Lambda_5}^{\mathrm{Com}(G_2 \times S_n)} \, C(\Lambda_4, \Lambda_5; \Lambda_3) \tag{5.24}$$

In the special case when the gauge group is U(1), so that the fields commute, $\Lambda_3(S_n)$ is the trivial representation. This forces $\Lambda_4(S_n) = \Lambda_5(S_n)$. The same thing applies when considering bosonic oscillators carrying indices of $G_1 \times G_2$.

6   Gauge invariant operators

We have organised n copies of the fundamental fields in terms of representations of the global symmetry group G.

$$\mathcal{O}^{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} = C_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau}^{\vec{m}} W_{m_1} \otimes W_{m_2} \otimes \cdots \otimes W_{m_n} \tag{6.1}$$

We now introduce the U(N) gauge group indices

$$(W_m)_j^i \tag{6.2}$$

and view the W_m as operators on the fundamental representation V_N of U(N). To form gauge-invariant operators we multiply these matrices together and then take products of traces organised by the symmetric group element $\alpha \in S_n$

$$\mathrm{tr}(\alpha \, W_{m_1} \otimes W_{m_2} \otimes \cdots \otimes W_{m_n}) = (W_{m_1})_{i_{\alpha(1)}}^{i_1} (W_{m_2})_{i_{\alpha(2)}}^{i_2} \cdots (W_{m_n})_{i_{\alpha(n)}}^{i_n} \tag{6.3}$$

where the trace is being taken in $V_N^{\otimes n}$. We can reorganise these in terms of representations R of U(N) using the Schur-Weyl dual S_n representation matrices $D_{ij}^R(\alpha)$

$$\frac{1}{n!} \sum_{\alpha \in S_n} D_{ij}^R(\alpha) \, \mathrm{tr}(\alpha \, W_{m_1} \otimes W_{m_2} \cdots \otimes W_{m_n}) \tag{6.4}$$

As a representation of U(N), R has at most N rows.

Combining the free S_n indices with an S_n Clebsch-Gordan coefficient gives a gauge invariant operator

$$\mathcal{O}^{\Lambda,M_\Lambda,\Lambda_1,\tau,R,\tau_{\Lambda_1,R}} = C^{\tau_{\Lambda_1,R}} {}_{m_{\Lambda_1}}^{\Lambda_1} {}_i^R {}_j^R \; C^{\vec{m}}_{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} \frac{1}{n!} \sum_{\alpha \in S_n} D^R_{ij}(\alpha) \operatorname{tr}(\alpha \, W_{m_1} \otimes \cdots \otimes W_{m_n})$$

$$= C^{\tau_{\Lambda_1,R}} {}_{m_{\Lambda_1}}^{\Lambda_1} {}_i^R {}_j^R \; \frac{1}{n!} \sum_{\alpha \in S_n} D^R_{ij}(\alpha) \operatorname{tr}\left(\alpha \, \mathcal{O}^{\Lambda,M_\Lambda,\Lambda_1,m_{\Lambda_1},\tau} \right) \tag{6.5}$$

We can invert the Clebschs to recover from these operators the basic gauge invariant operators in (6.3). This means that our new basis is complete. It also counts correctly at finite N, as demonstrated in the next section. Furthermore, following the methods of [15], it is fully diagonal in all its labels.

$$\left\langle \mathcal{O}^{\Lambda,M_\Lambda,\Lambda_1,\tau,R,\tau_{\Lambda_1,R}} \left(\mathcal{O}^\dagger \right)^{\Lambda',M'_{\Lambda'},\Lambda'_1,\tau',R',\tau'_{\Lambda'_1,R'}} \right\rangle$$

$$= \delta_{\Lambda\Lambda'} \delta_{M_\Lambda M'_{\Lambda'}} \delta_{\Lambda_1\Lambda'_1} \delta_{\tau\tau'} \; C^{\tau_{\Lambda_1,R}} {}_{m_{\Lambda_1}}^{\Lambda_1} {}_i^R {}_j^R \; C^{\tau'_{\Lambda'_1,R'}} {}_{m'_{\Lambda'_1}}^{\Lambda'_1} {}_k^{R'} {}_l^{R'}$$

$$\times \frac{1}{n!} \sum_{\alpha \in S_n} D^R_{ij}(\alpha) \frac{1}{n!} \sum_{\alpha' \in S_n} D^{R'}_{kl}(\alpha') \sum_{\sigma \in S_n} D^{\Lambda_1}_{m_{\Lambda_1} m'_{\Lambda'_1}}(\sigma) \operatorname{tr}(\alpha\sigma\alpha'\sigma^{-1})$$

$$= \delta_{\Lambda\Lambda'} \; \delta_{M_\Lambda M'_{\Lambda'}} \; \delta_{\Lambda_1\Lambda'_1} \; \delta_{\tau\tau'} \; \delta_{RR'} \; \delta_{\tau_{\Lambda_1,R} \, \tau'_{\Lambda'_1,R'}} \; \frac{n! d_{\Lambda_1}}{d_R^2} \operatorname{Dim} R \tag{6.6}$$

In the second line we have used the canonical covariant correlator (4.20). $\operatorname{Dim} R$ is the U(N) dimension of R.

6.1 Finite N counting

We show here that the operators defined in equation (6.5) count correctly for finite N. The finite N partition function is given in terms of the single letter partition function $f(\mathbf{x})$, for bosonic $\mathbf{x}$, by an integral over the U(N) matrix U [33, 34].

$$\mathcal{Z} = \int [dU] \, \exp\left\{ \sum \frac{1}{m} f(\mathbf{x}^m) \operatorname{tr}(U^\dagger)^m \operatorname{tr} U^m \right\} \tag{6.7}$$

$f(\mathbf{x})$ is the character of the fundamental representation V_F. For U(3) it is

$$f(\mathbf{x}^m) = x_1^m + x_2^m + x_3^m \tag{6.8}$$

and for SL(2) it is

$$f(\mathbf{x}^m) = \frac{q^m}{1 - q^m} \tag{6.9}$$

Now we perform the group integration for U(N) following [21] (see also [35]). If we expand out

$$\exp\left\{ \sum \frac{1}{m} f(\mathbf{x}^m) \operatorname{tr}(U^\dagger)^m \operatorname{tr} U^m \right\} \tag{6.10}$$

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we get

$$\sum_n \sum_{C_{\mathbf{i}} \in S_n} \prod_{j=1}^n \left(f(\mathbf{x}^j) \right)^{i_j} \frac{1}{j^{i_j} i_j!} \operatorname{tr}(C_{\mathbf{i}} U) \operatorname{tr}(C_{\mathbf{i}} U^\dagger) \qquad (6.11)$$

where $C_{\mathbf{i}}$ is a partition of n or a conjugacy class of S_n with i_1 1-cycles, i_2 2-cycles, ... i_n n-cycles. In $\frac{1}{j^{i_j} i_j!}$ the j^{i_j} comes from the $\frac{1}{m}$ in (6.10) and the $i_j!$ comes from $\exp(x) = \sum_k \frac{1}{k!} x^k$.

Using the identity $\operatorname{tr}(C_{\mathbf{i}} U) = \sum_{R(\mathrm{U}(N))} \chi_R(C_{\mathbf{i}}) \chi_R(U)$ and the group integral

$$\int [dU] \chi_R(U) \chi_{R'}(U^\dagger) = \delta_{RR'} \qquad (6.12)$$

we get the finite N partition function

$$\mathcal{Z} = \sum_n \sum_{R(\mathrm{U}(N))} \sum_{C_{\mathbf{i}} \in S_n} \prod_{j=1}^n \left(f(\mathbf{x}^j) \right)^{i_j} \frac{1}{j^{i_j} i_j!} \chi_R(C_{\mathbf{i}}) \chi_R(C_{\mathbf{i}}) \qquad (6.13)$$

Now if we treat $\mathbf{x}$ as a diagonal matrix (for U(3) we have (x_1, x_2, x_3) on the diagonal, for SL(2) we have $(q, q^2, q^3, \dots)$) and use

$$\prod_{j=1}^n \left(f(\mathbf{x}^j) \right)^{i_j} = \operatorname{tr}(C_{\mathbf{i}} \mathbf{x}) = \sum_{\Lambda_1(S_n)} \chi_{\Lambda_1}(C_{\mathbf{i}}) \chi_{\Lambda_1}(\mathbf{x}) \qquad (6.14)$$

then we get

$$\mathcal{Z} = \sum_n \sum_{R(\mathrm{U}(N))} \sum_{\Lambda_1(S_n)} \chi_{\Lambda_1}(\mathbf{x}) \, C(R, R, \Lambda_1) \qquad (6.15)$$

where $C(R, R, \Lambda_1)$ is the number of possible $\tau_{\Lambda_1, R}$ multiplicities in (6.5), i.e. the number of times Λ_1 appears in the symmetric group tensor product $R \otimes R$.[2] As representations of U(N), we only sum over Young diagrams R with at most N rows.

We have treated the global symmetry group here as $GL(\infty)$. A further decomposition into irreps. of G gives

$$V_{\Lambda_1}^{GL(\infty)} = \sum_\Lambda V_\Lambda^G \otimes V_{\Lambda, \Lambda_1} \qquad (6.16)$$

When we do this we finally see that the operators in (6.5) provide this counting.

$$\mathcal{Z} = \sum_n \sum_{R(\mathrm{U}(N))} \sum_{\Lambda(G)} \sum_{\Lambda_1(S_n)} d_{\Lambda, \Lambda_1} \, \chi_\Lambda(\mathbf{x}) \, C(R, R, \Lambda_1) \qquad (6.17)$$

where $\chi_\Lambda(\mathbf{x})$ is now a G character and d_{Λ, Λ_1} is the dimension of V_{Λ, Λ_1} labelled by the τ index in (6.5).

---

[2]$C(R, S, T) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \chi_S(\sigma) \chi_T(\sigma)$ and $\prod_{j=1}^n \frac{1}{j^{i_j} i_j!} = \frac{|C_{\mathbf{i}}|}{n!}$ where $|C_{\mathbf{i}}|$ is the size of the class $C_{\mathbf{i}}$.

7 Worldvolume excitations of giants and gauge invariant operators

7.1 Worldvolume excitations: review and comments

We review and comment on some results from [22] on the worldvolume excitations of half-BPS giant gravitons. Consider 3-brane giants expanding in the AdS^5. Use coordinates (t, v_1, v_2, v_3, v_4) for the AdS where we have a metric

$$ds^2 = -\left(1 + \sum_{k=1}^4 v_k^2\right)dt^2 + L^2\left(\delta_{ij} + \frac{v_i v_j}{(1 + \sum_k v_k^2)}\right)dv_i dv_j \tag{7.1}$$

L is the AdS_5 or S^5-radius. The S^5 can be described in analogous coordinates

$$ds^2 = L^2\left[\left(1 - \sum_{k=1}^4 y_k^2\right)d\phi^2 + \left(\delta_{ij} + \frac{y_i y_j}{1 - \sum_k y_k^2}\right)dy_i dy_j\right] \tag{7.2}$$

In global coordinates the AdS metric is

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{(1 + \frac{r^2}{L^2})} + r^2 d\Omega_3^2 \tag{7.3}$$

It is also useful to write the S^5 metric as

$$ds^2 = L^2(d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\Omega_3^2) \tag{7.4}$$

The AdS-giant graviton solution has

$$\begin{aligned}
\phi &= \omega_0 t \\
\omega_0 &= \frac{1}{L} \\
P_\phi &= N\left(\frac{r_0}{L}\right)^2
\end{aligned} \tag{7.5}$$

and the half-BPS property guarantees the energy is $E = \frac{P_\phi}{L}$. The brane worldvolume coordinates are $\tau, \sigma_1, \sigma_2, \sigma_3$. The coordinate τ is identified with the global time t. The $\sigma_1, \sigma_2, \sigma_3$ are identified with angles in AdS.

The fluctuations are expanded as

$$\begin{aligned}
r &= r_0 + \epsilon\ \delta r(\tau, \sigma_1, \sigma_2, \sigma_3) \\
\phi &= \omega_0\tau + \epsilon\ \delta\phi(\tau, \sigma_1, \sigma_2, \sigma_3) \\
y_k &= \epsilon\ \delta y_k(\tau, \sigma_1, \sigma_2, \sigma_3)
\end{aligned} \tag{7.6}$$

These perturbations are expanded in spherical harmonics.

$$\begin{aligned}
\delta r(\tau, \sigma_i) &= \tilde{\delta} r\ e^{-i\omega\tau} Y_l(\tau, \sigma_i) \\
\delta\phi(\tau, \sigma_i) &= \tilde{\delta}\phi\ e^{-i\omega\tau} Y_l(\tau, \sigma_i) \\
\delta y_k &= \tilde{\delta} y_k\ e^{-i\omega\tau} Y_l(\tau, \sigma_i)
\end{aligned} \tag{7.7}$$

The (ϕ, y_k) are coordinates for the sphere S^5. The Y_l are spherical harmonics on $S^3 \subset AdS_5$. They are symmetric traceless representations of SO(4). They have a quadratic Casimir $l(l+2)$ for the symmetric traceless representation of dimension $(l+1)^2$. The frequencies of these oscillations are calculated from the linearized equations of motion of the brane actions

$$S = S_{\text{DBI}} + S_{\text{CS}} \tag{7.8}$$

They lead (after a small simplification of expressions in [22]) to three solutions

$$
\begin{aligned}
\omega_- &= \frac{l}{L} \\
\omega_+ &= \frac{l+2}{L} \\
\omega &= \frac{l+1}{L}
\end{aligned}
\tag{7.9}
$$

The modes with frequencies $\omega_\pm$ are related to linear combinations $\tilde{\delta}r, \tilde{\delta}\phi$. The frequency ω is related to four modes $\tilde{\delta}v_m$ which transform in the fundamental of SO(4) in SO(6). It is very interesting that these are all integer multiples of the AdS-scale and approach $\omega = l/L$ in the large l limit. Note also that ω is the frequency for oscillations in t, the global time of AdS. The energies of the fluctuating giant gravitons are given by $E = \frac{n}{L} + \omega$ where n is the angular momentum of the background giant. The energy is related to scaling dimension in the dual CFT [3]. These energy spacings in integer units of $\frac{1}{L}$ are precisely the sort of spacings we get in free Yang Mills theory. Taking large angular momentum limits as a way to reach a classical regime where strong and weak coupling coupling can be compared directly is familiar from [38].

The $Y_{l,m}$ are representations of SO(4). Specifying the eigenvalues of the Cartan amounts to fixing two spins S_1, S_2. The SL(2) sector of gauge theory operators we considered, involving multitraces of $\partial_{1+i2}^S X^n$ corresponds to rotations in a fixed plane. This means that in each space of spherical harmonics of given l we are looking at a single state. Now if we consider a second quantization in the field theory of the branes, we would introduce a Fock space generator $\alpha_l^\dagger$ for each spherical harmonic. This has energy l/L above the background energy of the brane. General states look like

$$\alpha_1^{\dagger \ k_1} \alpha_2^{\dagger \ k_2} \cdots |0\rangle \tag{7.10}$$

The number of states at excitation energy k is the number of ways of writing $k = k_1 + 2k_2 + \cdots = \sum_i k_i l_i$ which is the number of partitions of k. When we restore the full SO(4) we have states of the form

$$\alpha_{l_1,m_1}^\dagger \alpha_{l_2,m_2}^\dagger \cdots |0\rangle \tag{7.11}$$

In this case it is useful to restrict attention to the symmetric traceless representations $[k]$ of SO(4) with excitation energy equal to k. In this case, the number of excited states of total energy $L_0 = n + k$ is again given by partitions of k. In the discussion below we will show that that there is an easy way to get these states from the gauge theory. In greater generality we should consider states of the form

$$\alpha_{l_1,m_1,I_1}^\dagger \alpha_{l_2,m_2,I_2}^\dagger \cdots |0\rangle \tag{7.12}$$

where I's are indices running from 1 to 6 which label the six eigenmodes built from (7.7). Four of these are in the fundamental of SO(4) $\subset$ SO(6). The fact that the excitation energies are spaced in units of $\frac{1}{L}$ (rather than in units of the brane size) was a bit of a surprise, discussed at length in [22]. An important point is that the kind of integer spacing in (7.9) is exactly what we have in free Yang Mills limit of the dual CFT. We will see below that this Fock space structure of orthogonal states emerges indeed from the construction of gauge invariant operators in the free dual Yang Mills theory. A connection between excited giant gravitons and the formulae for excitation energies (7.9) was made in [36]. The unravelling of the Fock space structure of giant graviton worldvolume field theory from gauge invariant operator counting given below is new.

7.2 Comparison to gauge invariant operators

We have constructed, in section 2, the lowest weights of the SL(2) sector by mapping states

$$A_{a_1}^\dagger A_{a_2}^\dagger \cdots A_{a_l}^\dagger |0\rangle \tag{7.13}$$

in an oscillator construction of SL(2) to gauge theory operators. The index a transforms in the hook representation $[n-1,1]$ of S_n. The $A^\dagger$'s are bosons so we are looking at the symmetric tensor product of the hook. These were constructed as lowest weight states generated by P_{1i} which forms part of the SO(4, 2) conformal algebra. These excitations correspond to exciting one spin inside AdS (for more details on the geometry of multiple spins see for example [37] in the context of spinning strings), hence to states of the form (7.10). After describing how to lift this to more general SO(4, 2) states, we will show that the counting in the case of single giants agrees with the bulk analysis reviewed above. Note for now that the above states transform in $Sym(V_H^{\otimes k})$ of S_n.

When we consider the full SO(4, 2) symmetry, we have additional generators K_λ forming the fundamental of SO(4). Correspondingly we have P_λ transforming in the fundamental of SO(4). When we consider lowest weight states annihilated by all the K_λ, we have states of the form

$$A_{a_1\lambda_1}^\dagger A_{a_2\lambda_2}^\dagger \cdots A_{a_k\lambda_k}^\dagger |0\rangle \tag{7.14}$$

Among these LWS are those transforming in the symmetric traceless representation of SO(4) associated with the symmetric Young diagram $[k]$ and with energy $L_0 = n + k$. As discussed in 5.5, these are a simple class of states which do not require projecting out of states due to the equations of motion, which require setting $P_\lambda P_\lambda$ to zero. Since the λ's are symmetrised, and the $A^\dagger$ are bosons, the indices $a_1, a_2, \ldots, a_l$ are symmetric, i.e. we have the symmetric k-fold tensor power of the hook representation $[n-1,1]$ of S_n. Orthonormal states in this sector are then written as

$$C_{\lambda_1\cdots\lambda_k}^{[k],M_{[k]}} C_{a_1\ldots a_k}^{\Lambda_1,[k],m_{\lambda_1},\tau'} A_{a_1\lambda_1}^\dagger A_{a_2\lambda_2}^\dagger \cdots A_{a_k\lambda_k}^\dagger |0\rangle \tag{7.15}$$

The first Clebsch's are for the symmetric traceless of SO(4) which are precisely the representations we discussed under (7.11). The second Clebsch have been discussed before in section 2. They decompose the $Sym(V_H^{\otimes k})$ into irreps. of Λ_1 of S_n. When we form gauge

invariant operators as in section 6 there are constraints relating Λ_1 to the U(N) Young diagram R which organises the traces. This representation R in the half-BPS case allows a map to the type of giant [8]. Young diagrams with a few (order 1) long (order N for example) columns map to sphere giants. Those with a few long rows map to AdS giants. Non-abelian worldvolume symmetries emerge when we have rows or columns of equal length. This map also works for open string excitations and there are elegant tests involving the counting of states which are sensitive to the presence of non-abelian symmetries [10, 11].

Consider Young diagrams of the form $R = [n]$ which correspond to single AdS giants of angular momentum n. Recall that the gauge invariant operators are labelled by $R, \Lambda_{n+k}, M, \Lambda_1, \tau, \tau_{\Lambda_1,R}$. R is a U(N) irrep. Λ_{n+k} is the lowest weight of the SL(2) which is completely determined by the excitation energy l. M labels states in Λ_{n+k}. Λ_1 is an irrep. of S_n. τ runs over the multiplicity of Λ_1 in the symmetric tensor product of the hook representation. $\tau_{\Lambda_1,R}$ runs from 1 to $C(R, R, \Lambda_1)$. For fixed R the multiplicity of LWS is

$$\sum_{\Lambda_1} C(R, R, \Lambda_1) Mult(\mathrm{Sym}(V_H^{\otimes k}), \Lambda_1) \tag{7.16}$$

By summing over states for fixed R we can get excited states of a fixed type of giant worldvolume. In particular we are interested in $R = [n]$. The inner tensor product of R with itself only contains the identity rep. $\Lambda_1 = [n]$. So the number of lowest weights at level k is just the multiplicity of $[n]$ in the symmetric tensor product of the hook. We have a generating function for this derived in section 3. The generating function including the descendants, is (using (3.7) or (3.11))

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^n)} \tag{7.17}$$

The coefficient of q^k is the number of partitions of k with no part bigger than n. Note that n is the number of boxes in the Young diagram describing the giant. For the semiclassical approximation of giant brane worldvolume to be valid, this is of order N^α (for α close to 1), k is the excitation on the brane worldvolume, which we are treating in a linearized approximation, so we certainly want that to be small compared to n. When k is smaller than n, the above just counts unrestricted partitions of k. This matches the counting of Fock space states in (7.15).

Hence, in the regime of interest, where k is much bigger than one (so we can expect GKP [38] type arguments to be valid) but smaller than the energy of the brane, the above counting of partitions of k is exactly what we are getting from quantizing a class of vibrations of the AdS giant. Using this emergence of Fock space structures from the counting of states in the tensor product of $\mathrm{Sym}(V_H^{\otimes k})$ we therefore find the correct counting of gauge theory operators which correspond to states of the form (7.10) and (7.11) with energy $L_0 = n + k$ and with a single spin k in the case (7.10) or with SO(4) representation $[k]$ for (7.11).

In fact we can also see where the six different species of oscillations could come from. In the above discussion we have been considering BPS giants built from Schur polynomials of $X = X_1 + iX_2$ and then perturbed by replacing X with derivatives P_λ acting on X, of the form $P_\lambda^* X$. We could also consider powers of P_λ acting on X_i (with $i = 3, 4, 5, 6$) replacing

the X. And finally we can consider powers of P_λ acting on $X^\dagger$ as the impurities. So in all we have six types of impurities $P_\lambda^* X, P_\lambda^* X^\dagger, P_\lambda^* X_i$. These correspond to six sets of gauge invariant operators matching states with the right energies of the form (7.10), which come, in the spacetime worldvolume analysis to exciting quanta of $\delta r, \delta v_m, \delta\phi$ excitations. Given the simplicity of ω_- we would expect that they correspond to the simplest construction in gauge theory, namely using $P_\lambda^* X$ impurities, which they match precisely in energy. If we consider the states in (7.12) and restrict to the case where all the impurities are of the same type and the SO(4) representation is $[k]$ with the energy being $E = n + k$, then the above discussion extends easily to give the corresponding gauge theory duals. A complete account of the case with mixed impurities will be left for the future.

7.3 Comments

There are many interesting extensions of the above discussion which could be considered. We have chosen the simplest R of the form $[n]$ which correspond to AdS giants. If we consider $R = [n_1, n_2]$ and sum over Λ_1 as in (7.16) this should correspond to excitations in spacetime of multiple-giants described by a U(2) (if $n_1 = n_2$) or U(1) × U(1) (if $n_1 \neq n_2$) worldvolume DBI gauge theory. A similar simple counting of states holds true for excitations of S-giants [22]. They will be associated to spherical harmonics of an SO(4) in the SO(6). So we expect that excitations in the gauge theory from the SO(6) sector should also have this kind of free field counting in an appropriate large angular momentum limit. The SO(4) ⊂ SO(4, 2) excitations considered in (7.15) also exist for $R = [1^n]$. They should correspond to excitations of sphere giants, but it is not obvious to us how a Fock space structure emerges from considering their motions in the transverse AdS. It will be interesting to clarify this puzzle.

Note that we are making here a comparison between zero coupling in Yang Mills to spacetime calculations dual to strong coupling Yang Mills. This works best for large angular momenta where l is large so that the frequencies can all be approximated by $\omega = l$, but smaller than n which is the large angular momentum of the giant. This gives a different context of excitations of giant gravitons, where the basic idea of large quantum numbers allowing strong to weak coupling comparisons [38] continues to apply. Here the parameters N, k, n are all large.

There have been earlier discussions of supersymmetric states obtained from the quantization of moduli spaces of giants and the comparison with gauge theory counting [39–42]. In the discussion above we have been interested in all the excitations in the free theory of a given half-BPS giant. A subset of these will be supersymmetric but a lot of the states will be non-supersymmetric. We expect that, in analogy with discussions of semiclassical strings [37, 38] appropriate limits of large quantum numbers can be used to compare non-supersymmetric states. The new technical ingredient in the above treatment is the use of a diagonal basis of gauge theory operators at finite N, where the label R allows the identification of the giant in question, and additional global symmetry labels help the map to objects in spacetime. The use of symmetric group data in organising the multiplicities of states for fixed R and fixed global symmetry quantum numbers shows the emergence, in the

limit of large n, of Fock space counting from properties of symmetric group decompositions such as $\mathrm{Sym}(V_H^{\otimes n})$. At finite n we have a cut-off Fock space.

8 One-loop mixing in $\mathcal{N} = 4$

In [43] the one-loop mixing of the Clebsch-Gordan basis introduced in [15] for the $G = \mathrm{U}(2)$ sector of $\mathcal{N} = 4$ super Yang-Mills was analysed. These operators only mix if the $\mathrm{U}(N)$ representations specifying their multi-trace structures are related by the repositioning of a single box of the Young diagram. Here we find the same result for the full $\mathrm{PSU}(2,2|4)$ sector, using our general characterisation of multi-trace operators with arbitrary global symmetry.

The complete one-loop non-planar dilatation operator is given by [26]

$$D(g) = D_0 - \frac{g_{\mathrm{YM}}^2}{8\pi^2}H + \mathcal{O}(g_{\mathrm{YM}}^3) \tag{8.1}$$

where

$$H = \sum_{j=0}^{\infty} h(j)(P_j)_{CD}^{AB} : \mathrm{tr}([W_A, \tilde{W}^C][W_B, \tilde{W}^D]) : \tag{8.2}$$

$(\tilde{W}^C)_j^i$ is the derivative $\frac{d}{d(W_C)_i^j}$. $h(j) \equiv \sum_{k=1}^{j} \frac{1}{k}$ are the harmonic numbers and P_j is the projector for $V_F \otimes V_F = \oplus_j V_j$. For SL(2) and $\mathrm{PSU}(2,2|4)$ V_j appears with unit multiplicity in $V_F^{\otimes 2}$ (cf. (3.3) where $m(j,2) = 1$).[3] The dilatation operator separates out $V_F^{\otimes 2}$ in $V_F^{\otimes n}$ and then projects onto it with the factors in (8.2).

The action of the dilatation operator has been analysed in the planar limit for single traces using the Bethe Ansatz (see for example [44, 45]). In the non-planar limit multi-trace operators can join and split [46]. We will find that the mixing is neatly constrained if we organise the multi-trace operators using $\mathrm{U}(N)$ representations as we have in (6.5).

The action of H on $\mathrm{tr}(\alpha W_{m_1} \cdots W_{m_n})$ is compactly written by introducing an extra index, tracing in V_N^{n+1} rather than V_N^n. The extra index encodes awkward contractions in the action of the dilatation operator.

$$: \mathrm{tr}([W_A, \tilde{W}^C][W_B, \tilde{W}^D]) : \mathrm{tr}(\alpha W_{m_1} \cdots W_{m_n}) =$$

$$\times \frac{1}{(n-2)!} \sum_{\sigma \in S_n} \delta_{m_{\sigma(n-1)}}^C \delta_{m_{\sigma(n)}}^D \sum_{\rho_1, \rho_2 \in S_{n+1}} f(\rho_1, \rho_2) \, \mathrm{tr}_{n+1}\Big(\rho_1 \sigma^{-1} \alpha \sigma \rho_2 W_{m_{\sigma(1)}} \cdots W_{m_{\sigma(n-2)}} W_A W_B \mathbb{I}_N\Big)$$

$\mathbb{I}_N$ is the $N \times N$ identity matrix. $f(\rho_1, \rho_2)$ is only non-zero on the S_3 subgroup of S_{n+1} that permutes the $n-1$ and n indices, where the derivatives act, and the new $n+1$ index. Its non-zero values give the four terms of the commutators in (8.2).

$$f(\,(n-1,n)\,,\,(n,n+1)\,) = 1$$
$$f(\,(n-1,n+1)\,,\,(n,n+1)\,) = -1$$
$$f(\,(n,n+1)\,,\,(n-1,n+1)\,) = -1$$
$$f(\,(n,n+1)\,,\,(n-1,n)\,) = 1 \tag{8.3}$$

---

[3]In the SL(2) $\times S_2$ decomposition of $V_F^{\otimes 2}$, the symmetric representation $V_{[2]}^{S_2}$ appears with even j and the antisymmetric $V_{[1,1]}^{S_2}$ with odd j.

If we introduce the projector we find

$$
\sum_{j=0}^{\infty} h(j)(P_j)_{CD}^{AB} : \mathrm{tr}([W_A, \tilde{W}^C][W_B, \tilde{W}^D]) : \mathrm{tr}(\alpha W_{m_1} \cdots W_{m_n}) = \sum_{\rho_1, \rho_2 \in S_{n+1}} f(\rho_1, \rho_2)
$$

$$
\times \frac{1}{(n-2)!} \sum_{\sigma \in S_n} \sum_{j=0}^{\infty} h(j) \, \mathrm{tr}_{n+1}\left(\rho_1 \sigma^{-1} \alpha \sigma \rho_2 W_{m_{\sigma(1)}} \cdots W_{m_{\sigma(n-2)}} P_j \left(W_{m_{\sigma(n-1)}} W_{m_{\sigma(n)}} \right) \mathbb{I}_N \right)
$$

Now consider the action on our gauge-invariant operator (6.5)

$$
H\mathcal{O}^{\Lambda, M_\Lambda, \Lambda_1, \tau, R, \tau_{\Lambda_1}, R} = \frac{1}{(n-2)!} \sum_{\rho_1, \rho_2 \in S_{n+1}} f(\rho_1, \rho_2) \, C^{\tau_{\Lambda_1}, R} \, {}_{m_{\Lambda_1}}^{\Lambda_1} \, {}_i^R \, {}_j^R \, \sum_{\alpha \in S_n} D_{ij}^R(\alpha)
$$

$$
\times \sum_{j=0}^{\infty} h(j) C^{\vec{m}}_{\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau} \, \mathrm{tr}_{n+1}\left(\rho_1 \alpha \rho_2 W_{m_1} \cdots W_{m_{n-2}} P_j \left(W_{m_{n-1}} W_{m_n} \right) \mathbb{I}_N \right) \qquad (8.4)
$$

Here, using properties of our operators, all the σ actions cancel.

To encapsulate the action of the projector we rewrite the covariant decomposition of $V_F^{\otimes n}$ in terms of $V_F^{\otimes n-2} \otimes V_F^{\otimes 2}$. We unclutter the notation by defining $|\mathbf{\Lambda}\rangle \equiv |\Lambda, M_\Lambda, \Lambda_1, m_{\Lambda_1}, \tau\rangle$ for the covariant basis.

$$
|\mathbf{\Lambda}\rangle = \sum_{\vec{m}} C^{\vec{m}}_{\mathbf{\Lambda}} \sum_{\mathbf{\Lambda}^{n-2}, \mathbf{\Lambda}^2} C^{\mathbf{\Lambda}^{n-2}}_{\vec{m}^{n-2}} \, C^{\mathbf{\Lambda}^2}_{\vec{m}^2} \, \left|\mathbf{\Lambda}^{n-2}\right\rangle \otimes \left|\mathbf{\Lambda}^2\right\rangle
$$

$$
= \sum_{\mathbf{\Lambda}^{n-2}, \mathbf{\Lambda}^2} \langle \mathbf{\Lambda}^{n-2}, \mathbf{\Lambda}^2 | \mathbf{\Lambda}\rangle \, \left|\mathbf{\Lambda}^{n-2}, \mathbf{\Lambda}^2\right\rangle \qquad (8.5)
$$

$\left|\mathbf{\Lambda}^{n-2}\right\rangle$ lives in $V_F^{\otimes n-2}$ while $\left|\mathbf{\Lambda}^2\right\rangle$ lives in $V_F^{\otimes 2}$. $\vec{m}^{n-2} = (m_1, \ldots, m_{n-2})$ and $\vec{m}^2 = (m_{n-1}, m_n)$.

The projector P_j in (8.4) projects onto $\mathbf{\Lambda}^2 = j$. The one-loop two-point function is then

$$
\left\langle (\mathcal{O}^\dagger)^{\Lambda', M'_{\Lambda'}, \Lambda'_1, \tau', R', \tau'_{\Lambda'_1}, R'} \, H \, \mathcal{O}^{\Lambda, M_\Lambda, \Lambda_1, \tau, R, \tau_{\Lambda_1}, R} \right\rangle
$$

$$
= \frac{1}{(n-2)!} \sum_{\rho_1, \rho_2 \in S_{n+1}} f(\rho_1, \rho_2) \, C^{\tau_{\Lambda_1}, R} \, {}_{m_{\Lambda_1}}^{\Lambda_1} \, {}_i^R \, {}_j^R \, C^{\tau'_{\Lambda'_1}, R'} \, {}_{m'_{\Lambda'_1}}^{\Lambda'_1} \, {}_k^{R'} \, {}_l^{R'} \, \sum_{\alpha, \alpha' \in S_n} D_{ij}^R(\alpha) \, D_{kl}^{R'}(\alpha')
$$

$$
\times \sum_{\mathbf{\Lambda}^{n-2}, \mathbf{\Lambda}^2 = j} h(j) \, \langle \mathbf{\Lambda}' | \mathbf{\Lambda}^{n-2}, \mathbf{\Lambda}^2 \rangle \, \langle \mathbf{\Lambda}^{n-2}, \mathbf{\Lambda}^2 | \mathbf{\Lambda}\rangle \, \mathrm{tr}_{n+1}\left(\rho_1 \alpha \rho_2 \alpha' \, \mathbb{I}_N^{n+1} \right) \qquad (8.6)
$$

The trace can be expressed as a sum over $(n+1)$-box representations T of S_{n+1} and $\mathrm{U}(N)$ with at most N rows.

$$
\mathrm{tr}_{n+1}\left(\rho_1 \alpha \rho_2 \alpha' \, \mathbb{I}_N^{n+1} \right) = \sum_{T \vdash n+1} \chi_T(\rho_1 \alpha \rho_2 \alpha') \mathrm{Dim} T \qquad (8.7)
$$

The α and α' sums in (8.6) force T to reduce to both R and R' for the S_n subgroup of S_{n+1}. Since T reduces on its S_n subgroup to those Young diagrams with a single box removed from T, R and R' must be related by the repositioning of a single box for this one-loop two-point function not to vanish. This analysis is pursued in more detail in [43].

The one-loop non-planar mixing of this complete basis of multi-trace operators is therefore highly constrained. Although the operators are not diagonal at one-loop, their very limited mixing suggests they are close to the eigenstates. It would be particularly interesting to find the sixteenth-BPS operators at one loop and gain an understanding of the counting of black hole entropy, along the lines of [42, 47].

9 Summary and outlook

We have given a general construction of diagonal gauge invariant operators for a $U(N)$ gauge theory with global symmetry G in the free field limit. The Clebsch-Gordan coefficients for the $G \times S_n$ decomposition of $V_F^{\otimes n}$, where V_F is a fundamental representation of G, play a crucial role. We have exploited the result of [15] which showed that once the covariant correlators are brought to a standard form, which we have called the "canonical covariant form" then the gauge invariant diagonalisation follows using Clebsch-Gordan coefficients of S_n. The $G \times S_n$ decomposition contains representation labels Λ, Λ_1 of G and S_n respectively. We showed in this paper how to use the corresponding Clebsch-Gordan coefficients, obeying standard Clebsch orthogonality properties (4.12), to construct operators with correlators of the canonical covariant form (4.20).

The construction of gauge invariant objects uses a representation label R corresponding to $U(N)$ and its Schur-Weyl dual S_n. This label appears in the simplest set-up in the half-BPS sector [8], and is interpreted in terms of giant gravitons. The final step of going from canonical covariant form to gauge invariant diagonal form uses the Clebsch-Gordan coefficients for the S_n inner tensor product $R \otimes R \to \Lambda_1$. We showed that the construction of a diagonal basis of gauge invariant operators matches the counting of gauge invariant operators done using Matrix Model techniques [21].

As special cases we have considered $G = U(M), SL(2), SO(6)$ which are relevant to specific sectors of $\mathcal{N} = 4$ SYM theory. In the case of $U(M)$ we have shown that the multi-matrix diagonalisation result of [15] contains a formula for the Clebsch-problem of $U(M)$ decomposition of $V_M^{\otimes n}$ in terms of the symmetric group data of branching coefficients. For $SL(2)$ we have shown how the $SL(2) \times S_n$ Clebsch-problem for the n-fold tensor product of the discrete series representation spanned by $X, \partial X, \partial^2 X \ldots$ can be solved by considering one energy level at a time, labelled by k, the total number of derivatives involved in the n-fold tensor product. The total number of lowest weight states appearing at fixed k can be neatly described in terms of oscillator constructions of $SL(2)$. This leads to a mapping of the problem of diagonalising the multiplicity of LWS at fixed k into a problem involving the $S_n \times S_k$ decomposition of the k-fold tensor power of a hook representation of S_n of dimension $n - 1$. This $S_n \times S_k$ problem has some surprising Fock space structures in the large n, k limit. These structures have been used to identify gauge invariant operators with energies and multiplicities matching those appearing in earlier work on the excitations of giant gravitons computed from the point of view of a worldvolume analysis [22].

We expect that further investigations on the diagonalisation of gauge invariant operators will allow more detailed comparisons between excitations of giant gravitons in spacetime and gauge theory operators. Comparisons going beyond free fields in gauge

theory and beyond the leading semiclassical approximations in giant gravitons will also be instructive. The formula for the 1-loop dilatation operator acting on our basis, given in section (8), is a step in this direction.

Acknowledgments

A   Clebsch-Gordan decomposition for $V_H^{\otimes k}$

We will collect several useful facts about the decomposition into $S_n \times S_k$ representations of the k-fold tensor product $V_H^{\otimes k}$ of the hook representation V_H of S_n associated with the Young diagram $[n-1, 1]$.

A.1   Multiplicities from characters

The k-fold tensor product decomposes as follows

$$V_H^{\otimes k} = \bigoplus_{\Lambda_1, \Lambda_2} V_{\Lambda_1} \otimes V_{\Lambda_2} \otimes V_{\Lambda_1, \Lambda_2} \tag{A.1}$$

Here V_{Λ_1} is an irrep of S_n, V_{Λ_2} is an irrep. of S_k , and V_{Λ_1, Λ_2} is an irrep of $\mathrm{Com}(S_n \times S_k)$, the algebra commuting with $S_n \times S_k$ in the $V_H^{\otimes k}$.

The dimensions d_{Λ_1, Λ_2} of V_{Λ_1, Λ_2} appear in the oscillator construction of LWS (lowest weight states) in the tensor products of the fundamental SL(2) representation. We can calculate these dimensions using characters of S_n and S_k.

$$d_{\Lambda_1, \Lambda_2} = tr_{V_H^{\otimes k}} \left(\frac{P_{\Lambda_1}}{d_{\Lambda_1}} \otimes \frac{P_{\Lambda_2}}{d_{\Lambda_2}} \right) \tag{A.2}$$

The projectors $P_{\Lambda_1}, P_{\Lambda_2}$ are given by

$$\frac{P_{\Lambda_1}}{d_{\Lambda_1}} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\Lambda_1}(\sigma) \sigma \tag{A.3}$$

and

$$\frac{P_{\Lambda_2}}{d_{\Lambda_2}} = \frac{1}{k!} \sum_{\tau \in S_k} \chi_{\Lambda_2}(\tau) \tau \tag{A.4}$$

Hence we can write

$$\begin{aligned} d_{\Lambda_1, \Lambda_2} &= \frac{1}{k!} \sum_{\tau \in S_k} \chi_{\Lambda_2}(\tau) \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\Lambda_1}(\sigma) tr_{V_H^{\otimes k}}(\tau \otimes \sigma) \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \chi_{\Lambda_2}(\tau) \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\Lambda_1}(\sigma) \prod_i (tr_{V_H}(\sigma^i))^{c_i(\tau)} \end{aligned} \tag{A.5}$$

For computer code to calculate this multiplicity see section C. Here $c_i(\tau)$ is the number of cycles of length i in the permutation τ. To see this note that

$$
\begin{aligned}
\langle a_1..a_k| \, \sigma \, |a_1..a_k\rangle &= \langle a_1..a_k|\sigma(a_1)\cdots\sigma(a_k)\rangle \\
\langle a_1..a_k| \, \tau \, |a_1..a_k\rangle &= \langle a_1..a_k|a_{\tau(1)}\cdots a_{\tau(k)}\rangle \\
\langle a_1| \, \sigma \, |a_2\rangle &= D^H_{a_1 a_2}(\sigma)
\end{aligned}
\tag{A.6}
$$

A.2 Clebsch-Gordan coefficients

For the Clebsch-decomposition of $V_H^{\otimes k}$ into $\Lambda_1(S_n) \otimes \Lambda_2(S_k)$, the first thing we need is the multiplicities. We also need in section 2 the properties of the Clebsch-Gordan coefficients for the case $\Lambda_2 = [k]$. Here we state some general properties valid for any Λ_2.

For the basic Clebsch problem of coupling a pair of irreps to a third $R \otimes S \to T$ we have previously used formulae of the type $DDD = CC$ and $DDC = DC$ derived for example in [48].[4] These were derived by inserting complete sets of states etc. Similar techniques lead to similar equations, which allow us , given the matrix elements of S_n irreps., to compute the Clebsch.

The following is the analog of $DDC = DC$

$$
\begin{aligned}
\sum_{a_1,\dots,a_k} D^H_{b_1 a_{\tau(1)}}(\sigma) D^H_{b_2 a_{\tau(2)}}(\sigma) &\cdots D^H_{b_k a_{\tau(k)}}(\sigma) C^{\Lambda_1,\Lambda_2,m_{\Lambda_1},m_{\Lambda_2};\ \tau_{\Lambda_1,\Lambda_2}}_{a_1\dots a_k} \\
&= C^{\Lambda_1,\Lambda_2,m'_{\Lambda_1},m'_{\Lambda_2};\ \tau_{\Lambda_1,\Lambda_2}}_{b_1\cdots b_k} D^{\Lambda_1}_{m'_{\Lambda_1} m_{\Lambda_1}}(\sigma) D^{\Lambda_2}_{m'_{\Lambda_2} m_{\Lambda_2}}(\tau)
\end{aligned}
\tag{A.7}
$$

The following is the analog of $DDD = CC$.

$$
\begin{aligned}
\sum_{\sigma \in S_n} \sum_{\tau \in S_k} D^{\Lambda_1}_{m_{\Lambda_1},m'_{\Lambda_1}}(\sigma) D^{\Lambda_2}_{m_{\Lambda_2},m'_{\Lambda_2}}(\tau) \quad & D^H_{a_1 b_{\tau(1)}}(\sigma)\cdots D^H_{a_k b_{\tau(k)}}(\sigma) \\
&= \sum_{\tau_{\Lambda_1,\Lambda_2}} C^{\tau_{\Lambda_1,\Lambda_2},\Lambda_1,\Lambda_2,m_{\Lambda_1},m_{\Lambda_2}}_{b_1\dots b_k} C^{\tau_{\Lambda_1,\Lambda_2},\Lambda_1,\Lambda_2,m'_{\Lambda_1},m'_{\Lambda_2}}_{a_1\dots a_k}
\end{aligned}
\tag{A.8}
$$

A.3 Symmetrised Clebsch from ordinary Clebsch

Now we specialise to give the properties for $\Lambda_2 = [k]$ which involve the symmetrised Clebsch. The symmetrised Clebsch-Gordan coefficients with the properties used in section 2 can be obtained from the Clebsch-Gordans for ordinary tensor products $V_H^{\otimes k}$. Consider states $|a_1, a_2 \cdots a_k\rangle$ in the tensor product. The action on the tensor product is

$$
\sigma|a_1,\dots,a_k\rangle = D^H_{b_1 a_1}(\sigma)\cdots D^H_{b_k a_k}(\sigma)|b_1,\dots,b_k\rangle
\tag{A.9}
$$

This action of $\sigma \in S_n$ commutes with S_k permutations which act as

$$
\alpha|a_1,\dots,a_k\rangle = |a_{\alpha(1)},\dots,a_{\alpha(k)}\rangle
\tag{A.10}
$$

The symmetric subspace of $V_H^{\otimes k}$ is isomorphic to the space of k oscillators $A_{a_1}\cdots A_{a_k}$. We can identify $A_{a_1}\cdots A_{a_k}$ with

$$
P_{\text{sym}}|a_1,\dots,a_k\rangle
\tag{A.11}
$$

---

[4] A minor notational point is that we are using the symbol C for Clebsch in this paper rather than S as in [15] and [48].

where the projector is $P_{\text{sym}} = \frac{1}{k!} \sum_{\alpha \in S_k} \alpha$ acting on $V_H^{\otimes k}$. The usual Clebsch decomposition gives the transformation matrix from the tensor product basis to a basis of irreps. $|\Lambda_1, i, \tau\rangle$ where τ is a multiplicity index and i is runs over the dimension d_{Λ_1}. Since P_{sym} commutes with S_n, its eigenvalues are constant on irreps, and since it is a projector they are 1 or 0. We can define the symmetric irreps Λ_1, i to be the set left invariant by P_{sym}. So we have

$$P_{\text{sym}}|\Lambda_1, i, \tau\rangle = |\Lambda_1, i, \tau\rangle \tag{A.12}$$

The Clebsch-Gordans restricted to the symmetric irreps are the symmetrised Clebsch-Gordans

$$
\begin{aligned}
C_{a_1 \cdots a_k}^{\Lambda_1, i, \tau} &\equiv \langle \Lambda_1, i, \tau | a_1 \cdots a_k \rangle \\
&= \langle \Lambda_1, i, \tau | P_{\text{sym}} | a_1 \cdots a_k \rangle
\end{aligned} \tag{A.13}
$$

If we permute the vectors in the tensor product we have

$$
\begin{aligned}
C_{a_{\alpha(1)} \cdots a_{\alpha(k)}}^{\Lambda_1, i, \tau} &= \langle \Lambda_1, i, \tau | P_{\text{sym}} \alpha | a_1 \cdots a_k \rangle \\
&= \langle \Lambda_1, i, \tau | P_{\text{sym}} | a_1 \cdots a_k \rangle \\
&= C_{a_1 \cdots a_k}^{\Lambda_1, i, \tau}
\end{aligned} \tag{A.14}
$$

The α can be absorbed in the redefinition of the summation over permutations in P_{sym}. This gives the desired symmetry of the symmetrised Clebsch (2.29).

Orthogonality (2.31) follows from the restriction of the usual orthogonality to the symmetric subspace. The identity (2.30) follows by considering

$$
\begin{aligned}
\langle \Lambda_1, i, \tau | \sigma | b_1, \ldots, b_k \rangle &= \langle \Lambda_1, i, \tau | P_{\text{sym}} \sigma | b_1, \ldots, b_k \rangle \\
&= \langle \Lambda_1, i, \tau | \sigma P_{\text{sym}} | b_1, \ldots, b_k \rangle
\end{aligned} \tag{A.15}
$$

and inserting on the left of σ a complete set of tensor product states or on the right of σ a complete set of symmetric irrep states.

A.4 Symmetrised Clebsch multiplicities

We give some examples of the symmetrised Clebsch multiplicities for $V_H = [n-1, 1]$. The tensor product of $V_H \otimes V_H$ is decomposed into irreps as $[n] + [n-1, 1] + [n-2, 2] + [n-2, 1^2]$. It can be checked that their dimensions add up to $(n-1)^2$. If we drop the last we get the dimensions adding to $\frac{n(n-1)}{2}$ as expected form the symmetric product. So we have

$$\text{Sym}(V_H \otimes V_H) = [n] + [n-1, 1] + [n-2, 2] \tag{A.16}$$

To get $\text{Sym}(V_H \otimes V_H \otimes V_H)$ we first take the above symmetric tensor product and then a further tensor product with $[n-1, 1]$. We need the tensor product

$$[n-2, 2] \otimes [n-1, 1] = [n-1, 1] + [n-2, 2] + [n-2, 1, 1] + [n-3, 3] + [n-3, 2, 1] \tag{A.17}$$

The dimensions add correctly. Then we need to implement the symmetric projection. One finds that the following gives the correct dimension counting

$$\text{Sym}(V_H \otimes V_H \otimes V_H) = [n] + 2[n-1, 1] + [n-2, 2] + [n-2, 1, 1] + [n-3, 3] \tag{A.18}$$

So the representation $[n-3, 2, 1]$ together with one of the two reps $[n-2, 1, 1]$, one of the two reps $[n-2, 2]$, and one of the three reps $[n-1, 1]$ have been projected out from $\text{Sym}(V_H \otimes V_H) \otimes V_H$.

B Action of S_n on $A_a^\dagger$ in detail

The orthogonal basis of $A_a^\dagger$ which we used in section 2 has the property that S_n acts on it via the standard Young-Yamanouchi orthogonal basis of the $[n-1,1]$ hook representation (as given for example in [48]). If $s_i = (i, i+1)$ are the 2-cycle permutations that generate S_n then we have

$$s_i A_a^\dagger = A_a^\dagger \quad \text{for } i \leq a-1 \text{ and } i \geq a+2$$
$$s_{i+1} A_i^\dagger = \frac{1}{i+1} A_i^\dagger + \frac{\sqrt{i(i+2)}}{i+1} A_{i+1}^\dagger$$
$$s_{i+1} A_{i+1}^\dagger = \frac{\sqrt{i(i+2)}}{i+1} A_i^\dagger - \frac{1}{i+1} A_{i+1}^\dagger \tag{B.1}$$

We can identify the representation of S_n formed by the A_a using general arguments. It is easy to see that there is no invariant vector under S_n, and that there is one invariant vector under S_{n-1} (namely A_{n-1}). The only irreducible representations of S_n which contain the invariant of S_{n-1} are $[n]$ and $[n-1,1]$. Having ruled out the symmetric irrep. $[n]$, the $(n-1)$ dimensional representation formed by the A_a can only be the irreducible $[n-1,1]$. More directly we can use the construction of the orthogonal representing matrices given in [48], which uses branching arguments.

C Code

Code written to calculate the various multiplicities discussed here is available under the GNU General Public Licence at http://www.nworbmot.org/code/. It is written in python for use with the SAGE open source computer algebra system.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [SPIRES].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109] [SPIRES].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [SPIRES].

[4] J.M. Maldacena and A. Strominger, *AdS$_3$ black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [hep-th/9804085] [SPIRES].

[5] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008 [hep-th/0003075] [SPIRES].

[6] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *JHEP* **08** (2000) 040 [hep-th/0008015] [SPIRES];
A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [hep-th/0008016] [SPIRES];

V. Balasubramanian, M. Berkooz, A. Naqvi and M.J. Strassler, *Giant gravitons in conformal field theory*, *JHEP* **04** (2002) 034 [hep-th/0107119] [SPIRES].

 [7] V. Balasubramanian, M. Berkooz, A. Naqvi and M.J. Strassler, *Giant gravitons in conformal field theory*, *JHEP* **04** (2002) 034 [hep-th/0107119] [SPIRES].

 [8] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual N = 4 SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222] [SPIRES].

 [9] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [hep-th/0403110] [SPIRES].

[10] V. Balasubramanian, D. Berenstein, B. Feng and M.-X. Huang, *D-branes in Yang-Mills theory and emergent gauge symmetry*, *JHEP* **03** (2005) 006 [hep-th/0411205] [SPIRES].

[11] R. de Mello Koch, J. Smolic and M. Smolic, *Giant gravitons — with strings attached (I)*, *JHEP* **06** (2007) 074 [hep-th/0701066] [SPIRES]; *Giant gravitons — with strings attached (II)*, *JHEP* **09** (2007) 049 [hep-th/0701067] [SPIRES];
D. Bekker, R. de Mello Koch and M. Stephanou, *Giant gravitons — with strings attached (III)*, *JHEP* **02** (2008) 029 [arXiv:0710.5372] [SPIRES].

[12] D. Berenstein, D.H. Correa and S.E. Vazquez, *A study of open strings ending on giant gravitons, spin chains and integrability*, *JHEP* **09** (2006) 065 [hep-th/0604123] [SPIRES].

[13] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174] [SPIRES].

[14] Y. Kimura and S. Ramgoolam, *Branes, anti-branes and Brauer algebras in gauge-gravity duality*, *JHEP* **11** (2007) 078 [arXiv:0709.2158] [SPIRES].

[15] T.W. Brown, P.J. Heslop and S. Ramgoolam, *Diagonal multi-matrix correlators and BPS operators in N = 4 SYM*, *JHEP* **02** (2008) 030 [arXiv:0711.0176] [SPIRES].

[16] R. Bhattacharyya, S. Collins and R.d.M. Koch, *Exact multi-matrix correlators*, *JHEP* **03** (2008) 044 [arXiv:0801.2061] [SPIRES];
R. Bhattacharyya, R. de Mello Koch and M. Stephanou, *Exact multi-restricted Schur polynomial correlators*, *JHEP* **06** (2008) 101 [arXiv:0805.3025] [SPIRES].

[17] T. Halverson, *Characters of the centralizer algebras of mixed tensor representations of $GL(r, \mathbb{C})$ and the quantum group $U_q(GL(r, \mathbb{C}))$*, *Pacific J. Math.* **174** (1996) 359.

[18] R. Goodman and N. Wallach, *Representations and Invariants of classical groups*, Cambridge University Press, Cambridge U.K. (1998).

[19] R.C. King, *Young tableaux, Schur functions and* SU(2) *plethysms*, *J. Phys.* **A 18** (1985) 2429.

[20] B. Feng, A. Hanany and Y.-H. He, *Counting gauge invariants: the plethystic program*, *JHEP* **03** (2007) 090 [hep-th/0701063] [SPIRES].

[21] S. Dutta and R. Gopakumar, *Free fermions and thermal AdS/CFT*, *JHEP* **03** (2008) 011 [arXiv:0711.0133] [SPIRES].

[22] S.R. Das, A. Jevicki and S.D. Mathur, *Vibration modes of giant gravitons*, *Phys. Rev.* **D 63** (2001) 024013 [hep-th/0009019] [SPIRES].

[23] P.H. Ginsparg, *Applied conformal field theory*, hep-th/9108028 [SPIRES].

[24] J. Polchinski, *String theory. Vol. 1: an introduction to the bosonic string*, Cambridge University Press, Cambridge U.K. (1998).

[25] T.W. Brown, R. de Mello Koch, S. Ramgoolam and N. Toumbas, *Correlators, probabilities and topologies in N = 4 SYM*, *JHEP* **03** (2007) 072 [hep-th/0611290] [SPIRES].

[26] N. Beisert, *The complete one-loop dilatation operator of N = 4 super Yang-Mills theory*, *Nucl. Phys.* **B 676** (2004) 3 [hep-th/0307015] [SPIRES].

[27] N. Beisert, M. Bianchi, J.F. Morales and H. Samtleben, *Higher spin symmetry and N = 4 SYM*, *JHEP* **07** (2004) 058 [hep-th/0405057] [SPIRES].

[28] S.E. Konstein, M.A. Vasiliev and V.N. Zaikin, *Conformal higher spin currents in any dimension and AdS/CFT correspondence*, *JHEP* **12** (2000) 018 [hep-th/0010239] [SPIRES].

[29] W. Fulton and J. Harris, *Representation theory: a first course*, Springer, Germany (1991).

[30] M. Bianchi and V. Didenko, *'Massive' higher spin multiplets and holography*, hep-th/0502220 [SPIRES].

[31] Koike and Terada, *Young diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n*, *J. Algebra* **107** (1987) 466.

[32] C. Grood, *Brauer algebras and Centralizer algebras for* SO(2n, ℂ), *J. Algebra* **222** (1999) 678.

[33] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The Hagedorn/deconfinement phase transition in weakly coupled large-N gauge theories*, *Adv. Theor. Math. Phys.* **8** (2004) 603 [hep-th/0310285] [SPIRES].

[34] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [hep-th/0510251] [SPIRES].

[35] F.A. Dolan, *Counting BPS operators in N = 4 SYM*, *Nucl. Phys.* **B 790** (2008) 432 [arXiv:0704.1038] [SPIRES].

[36] V. Balasubramanian, M.-X. Huang, T.S. Levi and A. Naqvi, *Open strings from N = 4 super Yang-Mills*, *JHEP* **08** (2002) 037 [hep-th/0204196] [SPIRES].

[37] B.J. Stefanski and A.A. Tseytlin, *Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations*, *JHEP* **05** (2004) 042 [hep-th/0404133] [SPIRES].

[38] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051] [SPIRES].

[39] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, *Supersymmetric states of N = 4 Yang-Mills from giant gravitons*, *JHEP* **12** (2007) 006 [hep-th/0606087] [SPIRES].

[40] G. Mandal and N.V. Suryanarayana, *Counting 1/8-BPS dual-giants*, *JHEP* **03** (2007) 031 [hep-th/0606088] [SPIRES].

[41] M.M. Caldarelli and P.J. Silva, *Multi giant graviton systems, SUSY breaking and CFT*, *JHEP* **02** (2004) 052 [hep-th/0401213] [SPIRES].

[42] L. Grant, P.A. Grassi, S. Kim and S. Minwalla, *Comments on 1/16 BPS quantum states and classical configurations*, *JHEP* **05** (2008) 049 [arXiv:0803.4183] [SPIRES].

[43] T.W. Brown, *Permutations and the loop*, *JHEP* **06** (2008) 008 [arXiv:0801.2094] [SPIRES].

[44] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for N = 4 super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208] [SPIRES].

[45] N. Beisert and M. Staudacher, *The N = 4 SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042] [SPIRES].

[46] S. Bellucci, P.Y. Casteill, J.F. Morales and C. Sochichiu, *Spin bit models from non-planar N = 4 SYM*, *Nucl. Phys.* **B 699** (2004) 151 [`hep-th/0404066`] [SPIRES].

[47] R.A. Janik and M. Trzetrzelewski, *Supergravitons from one loop perturbative N = 4 SYM*, *Phys. Rev.* **D 77** (2008) 085024 [`arXiv:0712.2714`] [SPIRES].

[48] M. Hamermesh, *Group theory and its applications to physical problems*, Addison-Wesley Publishing Company, U.S.A. (1962).